Network Television Dynamics: A Conceptual Mathematical Model

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Abstract

In this thesis we will be expanding on modifications we previously made relating to Edwards and Buckmire's model of box office dynamics to network television. We will introduce the number of viewers with a negative reaction to the product as a function with respect to time. The interaction of the main parameters of the box office dynamics as they translate to the network television problem: viewership, revenue, and audience perception for television programming will be presented as a conceptual model of a system of three 1st order differential equations. The eigenvalue method, Routh-Hurwitz stability criterion, and control theory will be used to solve the problem and the stability of the solution will be checked. Finally, based on the numerical solution and its stability, recommendations will be presented.

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1 Introduction

Television is at heart a number's game, and both the critical acclaim for the program and the fame of those involved in the production are irrelevant if the show cannot garner the ratings to support it. The industry is littered with the remains of shows which were heavily touted as the next great thing, many with the backing of some of the leading names in the entertainment industry, that failed to finish their initial season before being canceled. This is especially true for broadcast television which, unlike cable or premium television content, relies almost exclusively on the sale of advertising during its programming. It is of critical importance for these networks to quickly identify which of their shows are faltering and which are succeeding in order to make decisions regarding their programming line-up, in order to ensure an optimal revenue stream.

In film the product does not change from week to week, so the percentage of viewers with a negative reaction remains fairly stable. This makes the total number of viewers with a negative reaction is a strictly increasing function of time. In television, however, shows can and frequently do improve the longer they run. Accordingly, the percentage of people with a negative response to the film, which was treated as a parameter in Edwards and Buckmire's original model, may fluctuate up or down from week to week depending on how the previous episode was received and therefore need to be modeled to account for this trait present in television programming. In adjusting the model to network television, it is necessary to identify the relevant parameters and how they affect the three functions that are modeled. Moreover, given that the expected result is a system of coupled ordinary differential equations, the question of how the functions

interact with each other, and what are the conditions necessary to ensure the stability of the system remain.

2 Literature Review

Most existing models for the behavior of audiences rely on probabilistic and statistical methods. We will now go over some prominent models for determining the number of viewers of visual media, and characterize the principles.

A two-sided model of the relationship between viewer and advertisers demands, using a discrete choice model to gauge viewer demand for a television program versus the advertisers demand for ad space during that program was proposed in [1]. This model focuses on the demand for the television show, but does not directly address the network's revenue or the number of viewers for specific programs. A Bayesian statistical method with a model of choice with a Tobit model to examine the affect the opinions of others has on viewership was examined in [2], however this model has simplistic dynamics forced by the so-called Aumman Agreement Theorem. The use of ANOVA hypothesis testing to examine viewers' satisfaction relating to television dramas was given in [3], but this method leaves the conceptual structure of viewer opinion and. therefore, long run behavior unclear. The measure of similarity as a probabilistic method of modeling viewer preference and choice, with iso-utility contours and clusterwise logit regression was analyzed in [4] but like the simpler ANOVA testing this does not give an idea of the dynamics of the preferences.

In lieu of probabilistic methods some models have used attributing basis with bifurcated results in a purchasing profile framework, to which they applied cluster and regression analysis to model advertising price which relates to revenue [5]. This analysis explored the relationship between the behavior of viewers as consumers and its effect on determining ad price and thus ad revenue; however it does not attempt to examine the

effect of negative response to the programming on ad price or revenue, nor does it model the effect of negative reaction in determining the number of viewers . A Heuristic method based on the audience flow model to maximize audience share can be found in [6], but this approach lacks a solid foundation. The author chose a sample size small enough that they could run all possible scheduling permutations, and then analyzed the results to find the optimal programming schedule. This is essentially a form of guess and check, and is impractical for use on a larger scale.

There are a few models that use deterministic methods in other sectors of the entertainment industry, including Enomoto and Ghosh's model of pricing home-video releases of motion pictures [7]; Edwards and Buckmire's deterministic model [8] created as a predictive tool for the behavior of motion pictures. At the time of publication they noted that most existing models for this field relied on probabilistic methods.

Though the majority of existing models based on discrete statistical methods, a deterministic model provides a new method of examining the relationship between viewership, revenue, and audience perception for television programming. Additionally, network schedulers would have access to multiple methodologies for analyzing and predicting the success of the programing lineup, and maximizing their revenue stream.

3 Mathematical Model

Edwards and Buckmire were pioneers in introducing a deterministic ODE model for the motion picture industry. In Edwards and Buckmire's model the rates, $\frac{dS}{dt}$ and $\frac{dA}{dt}$, measured the change in the number of screens a film is shown on (S) and the average revenue per screen (A), and are in combination representative of the number of viewers per week for the film being analyzed. In the television industry, ad revenue is calculated in terms of the number of viewers of a show on a given network. While more stations equates to a larger number of potential viewers, the number of stations is not directly used in calculating ad prices. Accordingly we will consider the number of viewers to be modeled by a single equation, and a third equation modeling the number of viewers with a negative reaction will be added to the model, in addition to the equation for cumulative gross revenue.

The following parameters from the film model will not be featured in the television model: Ticket price (P), that is the monetary cost required to view the entertainment, is not included in the model as every network television program has the same monetary cost to the viewers it is available to, so that price differentiation is non-existent. The number of times the average viewer watches a specific episode (D) is irrelevant to advertisers and is not measured. As for the number of times the average viewer watches a show in a season, repeat viewing over an entire season is accounted for by our model.

The following system of three first order differential equations models the interaction among cumulative gross revenue from ad sales (G), number of viewers (V), and the number of individuals with a negative response to the program (H);

$$\begin{cases} \dfrac{dG}{dt} = \dfrac{\mathrm{V}}{1000}\left[1 - P_H\left(\dfrac{H}{V} - H_{\%}\right)\right] cpm * (\varphi) & ;\ G(0) = G_0 \quad (3.1) \\ \dfrac{dV}{dt} = -\dfrac{\alpha_V}{E_{\%}(1+\epsilon)}\left[\dfrac{\psi + C}{1+\gamma M}V - \beta G\left(\dfrac{H}{V}\right)^2\right] & ;\ V(0) = V_0 \quad (3.2) \\ \dfrac{dH}{dt} = \dfrac{\alpha_H}{1+\epsilon}\left(\dfrac{1-\mathcal{S}}{1+\gamma M} + \beta\right)H - qP_G GH & ;\ H(0) = H_0 \quad (3.3) \end{cases}$$

Where its variables and parameters are summarized in Table 1 and Table 2.

Table 1: Variables & Functions

$\boldsymbol{t}$	$time\ in\ weeks$
$\boldsymbol{G}$	$cumulative\ gross\ revenue\ from\ ad\ sales$
$\boldsymbol{V}$	$number\ of\ viewers$
$\boldsymbol{H}$	$number\ of\ viewers\ with\ a\ negative\ reaction\ to\ program$

Table 2: Parameters

$\boldsymbol{\varphi}$	$number\ of\ ad\ units\ for\ show$
$\boldsymbol{cpm}$	$price\ in\ \$\ for\ each\ unit\ of\ ad\ time, per\ 1000\ viewers$
$\boldsymbol{P_H}$	$rate\ at\ which\ negative\ perceptions\ reduce\ revenue$
$\boldsymbol{H_{\%}}$	$tolerance\ for\ negative\ reactions\ as\ a\ percentage\ of\ viewers$
$\boldsymbol{\alpha_V}$	$rate\ of\ decay\ for\ viewers$
$\boldsymbol{E_{\%}}$	$average\ number\ of\ episodes\ watched\ per\ season$
$\boldsymbol{\beta}$	$effectivness\ of\ negative\ word\ of\ mouth$
$\boldsymbol{\gamma}$	$effectiveness\ of\ advertising$
$\boldsymbol{\varepsilon}$	$effectiveness\ of\ critical\ reviews$
$\boldsymbol{M}$	$marketing\ budget$
$\boldsymbol{C}$	$negative\ effect\ of\ the\ popularity\ of\ competing\ shows\ on\ viewership$
$\boldsymbol{\psi}$	$effect\ of\ timeslot\ changes\ or\ disruptions\ on\ viewership$
$\boldsymbol{\alpha_H}$	$rate\ of\ growth\ of\ individuals\ with\ negative\ reaction\ to\ program$
$\boldsymbol{\mathcal{S}}$	$social\ perceptions\ influencing\ change\ in\ reactions\ to\ program$
$\boldsymbol{q}$	$measure\ of\ percieved\ quality\ of\ program\ by\ viewers$
$\boldsymbol{P_G}$	$percent\ of\ revenue\ reinvested\ into\ improving\ program$

3.1 Gross Revenue

It is necessary to consider that networks only receive ad revenue while the television program is broadcasting. In reality, a major revenue stream for cable networks is subscription costs, which have a complex relationship with viewership and are therefore omitted in most analyses. By focusing exclusively on network, or broadcast, television we can circumvent the need to include this. Some shows may air for a single season, successful ones may air for a few years, or even in exceptional cases decades. Nevertheless, a program's lifespan is finite in nature and thus ad revenue cannot be generated in perpetuity. Eventually no further ad revenue will be generated for the network which produced it. Accordingly, $\lim_{t\to\infty} \frac{dG}{dt} = 0$ which implies the function $G(t)$ must be logistic in nature, and $\frac{dG}{dt}$ may then be expected to take a form similar to $\frac{dG}{dt} = aV - bV^2$, where a and b are some constants derived from relevant parameters.

When considering the cumulative gross revenue G for network television, its first derivative $\frac{dG}{dt}$ is, at its simplest, merely the ad revenue generated at time t. Relevant to the consideration of the cumulative gross revenue, is the number of units of advertising that are sold for the program (φ), and the price for each unit sold. Advertising is typically sold in 15/30/60 second slots; for the sake of calculations we will assume an average time of 30 seconds per ad with approximately 15 minutes of ad time per one hour show. Prices for television ads are given in cost per mille (*cpm*), this is the price per 1000 viewers for a unit of ad time. Thus $\frac{dG}{dt}$ is a product of the number of ad units sold and the price per unit, giving us the first order ODE $\frac{dG}{dt} = \frac{V}{1000}\varphi * cpm$. This equation relies on the assumption that the reaction of viewers to the program has no effect on ad revenue. This assumption

is unrealistic in nature, as extreme negative responses to programming may cause advertisers to pull advertising support from the program [9]. Then some amount of revenue is lost proportional to H and V, accordingly $\frac{dG}{dt}$ takes the form $\frac{V}{1000}(1 - bH)\varphi * cpm$. The second term fulfills a similar function to the V^2 term, as H may be considered as a subset of V. However, this alteration to the model assumes that any negative response will result in a loss of revenue. This is a problematic assumption, as no show can possibly attain a universally positive audience response. Accordingly, if some negative response to a program is expected, then it is only in extreme cases where the level of negative reaction exceeds some predefined threshold that this has a negative effect on revenue. Thus the negative effect with respect to H can be thought of as $(H - H_T)$, where H_T represents the level of negative responses which the network considers acceptable. This can be expected to vary among television networks, dependent on target audience and brand perception. With some networks having a lower tolerance to negative audience reception, while others have a high tolerance for controversial programming and may even consider it an integral part of their identity. For calculation purposes, it is more useful to think of the H_T as a percentage of the population, in which case we get:

$$\frac{dG}{dt} = \frac{V}{1000}\left[1 - b\left(\frac{H}{V} - H_{\%}\right)\right]\varphi * cpm.$$

3.2 Viewership

Implied by the asymptotic limit of G, is the fact that by its nature the number of viewers must peak at some point. At that point the market will become saturated and lose viewers until there can be no further revenue generated. A highly successful show may take years to reach this point, and may even be able to sustain it for a period of time, but eventually the viewing audience will move on to the next trend and the show will begin to lose viewers. This loss of viewers will continue until, at least in the model, the number of viewers reaches zero. In reality, the more likely scenario is that the number of viewers will decrease until it reaches some critical threshold and it is canceled by the network's programming director. From this, we can infer that the $\lim_{t\to\infty} V(t) = 0$ which implies that, absent any other considerations, the rate of change of viewers takes the form $\frac{dV}{dt} = -\alpha_V V$. Where α_V is the rate of decay, which is determined by casting, genre, and other predetermined factors relevant to the program. This form of $\frac{dV}{dt}$ is based on six assumptions;

i) A viewer, having watched one episode, will not return subsequently to watch any additional episodes.

The entire premise of network television is to keep viewers returning to the programming for each episode. Now not every viewer will watch every single episode, so the average percentage of episodes watched per season must be estimated. Furthermore, programs may air in one of several season formats; the standard long season typically has 22-24 episodes, a short summer or midseason replacement program will

generally consist of 13 episodes, and a miniseries may have between two and ten episodes. Dividing the decay rate of V by the product of the average percentage of episodes watched for the genre and network and the number of episodes in the season for the show being analyzed will slow the rate at which the show loses viewers, and gives $\frac{dV}{dt} = -\frac{\alpha_V}{E_{\%}} V$.

ii) The television network makes no effort to promote their programming.

This is clearly not the case, networks can and generally do spend money on various forms of marketing to promote their programs both before and after the series premier. Therefore, the effect of this must be accounted for both in the amount spent (M) and the effectiveness (γ) of the marketing campaign. Similar to repeat viewing, a successful marketing should slow the rate of decay for V. Thus $\frac{dV}{dt}$ will take the form $-\frac{\alpha_V}{E_{\%}(1+\gamma M)} V$.

iii) Critical reviews and the opinions of others have no effect on viewership.

Both critical reviews and the opinions of those around them will affect viewing behavior. The opinions of reviewers (ε) may either increase or decrease the rate of decay. Additionally, word of mouth, the opinions of those in the viewer's social sphere, may also affect viewership and upgrade $\frac{dV}{dt}$ to $-\frac{\alpha_V}{E_{\%}(1+\epsilon)}\left(\frac{1}{(1+\gamma M)} - \beta\right) V$

iv) Schedule changes and programming decisions of other networks have no effect on viewership.

Unlike movies TV shows must compete with other shows that air at the same time on different stations. Since the viewer cannot effectively watch two shows simultaneously, this creates a forced choice where the viewer must decide which shows are most appealing to them. The relative popularity and similarity/difference in genres of these other programs has an impact on the viewership for the show being analyzed. Time delayed viewing (DVRs, video on demand, online streaming, etc.) has provided some relief from this choice allowing viewers to watch their 2nd or even 3rd choice shows at a later date, time permitting; however, it cannot be assumed that all viewers will have access to this or that they will necessarily be sufficiently interested in the other shows to go to the effort of seeking them out at a later date. Accordingly, the effect of the popularity of competing programming (C) must be considered as a parameter of particular importance to network television.

The effect of timeslot changes or disruptions in broadcast (ψ) is a parameter that is unique to the television industry, and must be considered in the model. Networks may choose to change the timeslot of a particular show; that is, change the date and time when the show is broadcast. If the network changes the time slot they may lose viewers for several reasons. The new timeslot may place the show in competition with another show that the viewer prefers, or if the viewer is unaware of the change they may not know where the show has moved to or else they may assume that the show has been canceled. Alternatively, a change in timeslot may prove beneficial to a struggling show by moving

it out of competition with a highly successful competitor, or placing it at a historically more advantageous timeslot (i.e. Monday nights) [10].

Networks may also preempt a show and air other programming in its place, due to unexpected news coverage, or special events such as sporting events or movies. Regardless of the reason, this can have a negative impact on viewership. If these events occur too frequently the viewer may forget about, lose interest in, or get out of the habit of watching that particular program.

The effect of these parameters on viewership may be enhanced or ameliorated by effective marketing, then $\frac{dV}{dt}$ can be upgraded to $-\frac{\alpha_V}{E_{\%}(1+\epsilon)}\left(\frac{\psi+C}{(1+\gamma M)}-\beta\right)V$

v) The network does not reinvest profits into improving viewership.

Networks will redirect money from underperforming productions to increase a successful show's budget. $\frac{dV}{dt}=-\frac{\alpha_V}{E_{\%}(1+\epsilon)}\left(\frac{\psi+C}{(1+\gamma M)}-\beta G\right)V$, this form is flawed as G will tend to be substantially larger than V and βG will tend to override the other terms. A manner of scaling this must be introduced.

vi) There are no negative reactions to the program, or the negative reactions have no effect on viewership.

There will naturally be some negative response to even the most popular of programming, and this negative response must have some effect on viewership. If β is a measure of the effectiveness of word of mouth, then βH will account for the word of mouth from those with a negative reaction to the program, by considering negative

responses as a percentage of viewers and then squaring the term will act as a control for the relative differences in scale between V and G.

Then, $\frac{dV}{dt} = \frac{-\alpha_V}{E_{\%}(1+\epsilon)}\left[\frac{1}{(1+\gamma M)}V - \beta G\left(\frac{H}{V}\right)^2\right]$ will be the second equation in our model.

3.3 Negative Response

In the film industry a given product does not change from week to week, and therefore the percentage of viewers with a negative reaction to the motion picture remains fairly stable; correspondingly the total number of viewers with a negative reaction can be modeled as an increasing function H(t) and is considered as a parameter in the original model. In television however, shows are frequently perceived by audiences to change in quality over the lifespan of the program. Accordingly, H(t) may fluctuate from week to week depending on how the previous episode was received, and is more accurately considered as functions with respect to time for television programs.

In the film model H(t) is assumed to increase at a given rate over the films lifespan. In the absence of any factors specific to the television industry; $\frac{dH}{dt} = \alpha_H H$, this is based on four assumptions.

i) Critical reviews and word of mouth have no effect on people's reactions to the program.

Similar to our work on $\frac{dV}{dt}$, $\frac{dH}{dt}$ will take the form $\frac{\alpha_V}{1+\varepsilon}[1+\beta]H$

ii) No one ever changes their opinion about the program, once they have developed a negative opinion about the program they will continue to hold that opinion.

People can and do change their opinions. This may be especially relevant for television shows because the content changes from episode to episode. Changes in casting, production quality, and current events can all cause someone to change their opinion regarding the show. Hence, $\frac{dH}{dt}$ can be upgraded to $\frac{\alpha_H}{1+\varepsilon}(1+\beta)H - \Delta H$, where Δ represent factors such as perceived quality of the production (q), and social factors (S) affecting public perception of the program, and $\frac{dH}{dt} = \frac{\alpha_H}{1+\varepsilon}(1+\beta)H - (\mathrm{S}+\mathrm{q})H$.

iii) The network does not reinvest profits into improving the quality of the show.

Networks will reinvest a portion of their revenue to increase a successful show's budget, which can result in improved production values, better writers, and allow the show to retain or replace actors; Hence $\frac{dH}{dt}$ can improve to:

$$\left[\frac{\alpha_H}{1+\varepsilon}(1+\beta) - \mathrm{S} - \mathrm{q}P_G G\right] H$$

iv) Marketing has no effect on people's perception of the show.

The primary purpose of marketing is to sway opinions. Marketing will primarily affect social factors and the base rate of growth for H, thus the third equation will take the form: $\frac{dH}{dt} = \left[\frac{\alpha_V}{1+\varepsilon}\left(\frac{1}{1+\gamma M} + \beta\right) - \frac{\mathrm{S}}{1+\gamma M}\right]\mathrm{H} - \mathrm{q}P_G GH$

4 Stability Analysis

A system of equations is considered stable when a finite input does not result in an infinite value from any of the functions in the system. That is to say when the limit as t approaches infinity exists and is a real number. For a system of ordinary differential equations, it is sufficient for the real part of all eigenvalues to be less than zero in order to guarantee the stability of the system.

For the purpose of analyzing the stability of the system, as the parameters are all real numbers and any combination of them is also a real number (Table 3), then the system may be simplified as:

$$\begin{cases} \dfrac{dG}{dt} = P_1 V - P_2 \left(\dfrac{H}{V} - P_3\right) \mathrm{V} & (4.1) \\ \dfrac{dV}{dt} = -P_4 V + P_5 G \left(\dfrac{H}{V}\right)^2 & (4.2) \\ \dfrac{dH}{dt} = P_6 H - P_7 G H & (4.3) \end{cases}$$

Table 3: Simplified Parameters

$\boldsymbol{P_1}$	$\dfrac{cpm * (\varphi)}{1000}$
$\boldsymbol{P_2}$	$\dfrac{cpm * (\varphi)}{1000} P_H$
$\boldsymbol{P_3}$	$H_{\%}$
$\boldsymbol{P_4}$	$\left(\dfrac{\alpha_V}{E_{\%}(1+\epsilon)}\right)\left(\dfrac{\psi + C}{1+\gamma M}\right)$
$\boldsymbol{P_5}$	$\left(\dfrac{\alpha_V}{E_{\%}(1+\epsilon)}\right)\beta$
$\boldsymbol{P_6}$	$\left(\dfrac{\alpha_H}{1+\epsilon}\right)\left(\dfrac{1-\mathcal{S}}{1+\gamma M}\right)$
$\boldsymbol{P_7}$	qP_G

For the purpose of analysis some values of P will be approximated where reasonable values may be inferred, this will be discussed in more detail in section 4.2.

4.1 Numerical Stability

While some of the parameters have values which may be inferred from existing data, others rely on measures of public perception that are not currently studied numerically. Accordingly, of the simplified coefficients in equation 4.1 – 4.3, P_4, P_5 and P_6 cannot be estimated from currently available data. We will use those values that can be estimated and variations of the three that cannot, to see how the system behaves. For this we will use various values of P_4, P_5, and P_6, and additionally we will examine the effect of the ratio $\frac{H_0}{V_0}$ on the stability of the system.

4.1.1 Variation of Parameters

To begin we will consider the stability of the system when $P_4 = P_5 = P_6 = 1$. It is apparent from Figure 1, that the gross revenue function is unstable over large intervals of t. Even so, the system may be stable over a smaller interval of t, that is of sufficient size to model a single television season, as in Figure 2. In this instance, the overall instability of the system does not preclude the use of the model over finite periods of time.

Figure 2: $\boldsymbol{P_4 = P_5 = P_6 = 1}$**, over** $\boldsymbol{t \in (0, 100)}$

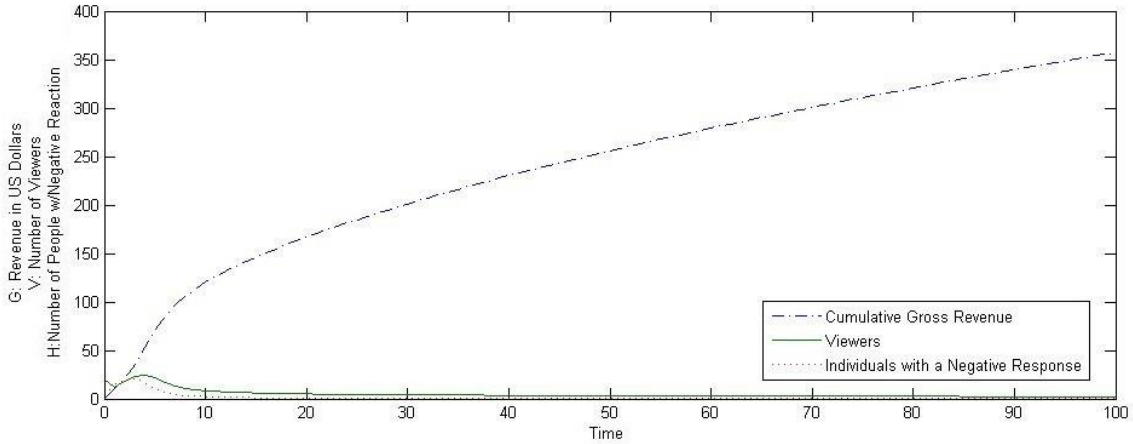
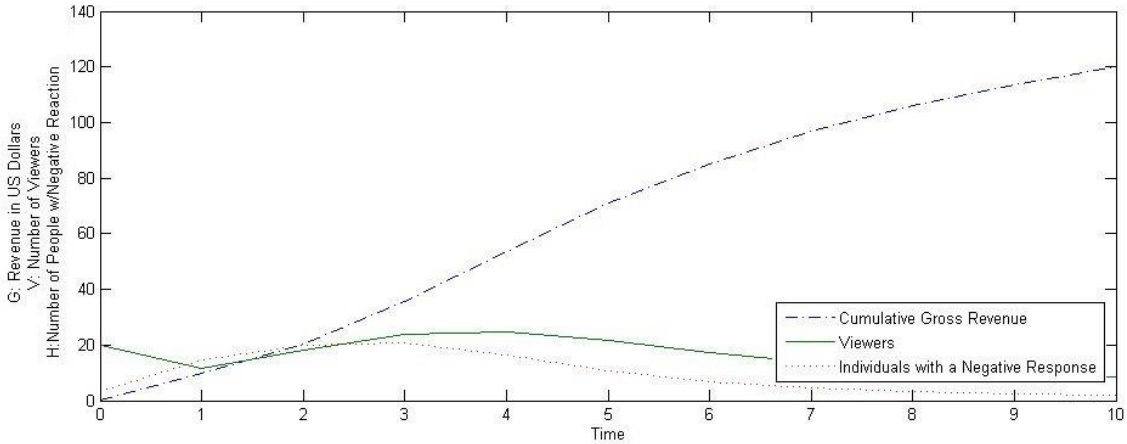


Figure 3: $\boldsymbol{P_4 = P_5 = P_6 = 1}$**, over** $\boldsymbol{t \in (0, 10)}$



Next we consider increasing the size of our parameters by an order of magnitude, so that $P_4 = P_5 = P_6 = 10$. We expect this will increase the instability of our system, and thus will decrease the interval over t for which our model is considered functionally stable. We can see, (Figure 3) that not only is our system now unstable in the interval of stability for $P_4 = P_5 = P_6 = 1$, but the number of individuals with a negative reaction exceeds the number of viewers much of the first third of the graph. Indeed, it is necessary to reduce the interval by a third (Figure 4) to find an interval over which the gross revenue function can be considered functionally stable, and even then those with a negative reaction exceed the number of viewers for most of the graph. Undoubtedly a situation most television networks would find intolerable.

This type of behavior is perhaps indicative of a show that is either poorly produced, or has some controversy surrounding it to the point where people develop and retain a poor opinion of it even when they no longer, or perhaps have never, watched it. The eventual decay of H, until it is less than V, is consistent with improvement in production quality or with the show successfully distancing itself from the source of negative responses.

Figure 4: $P_4 = P_5 = P_6 = 10$, over $t \in (0, 10)$

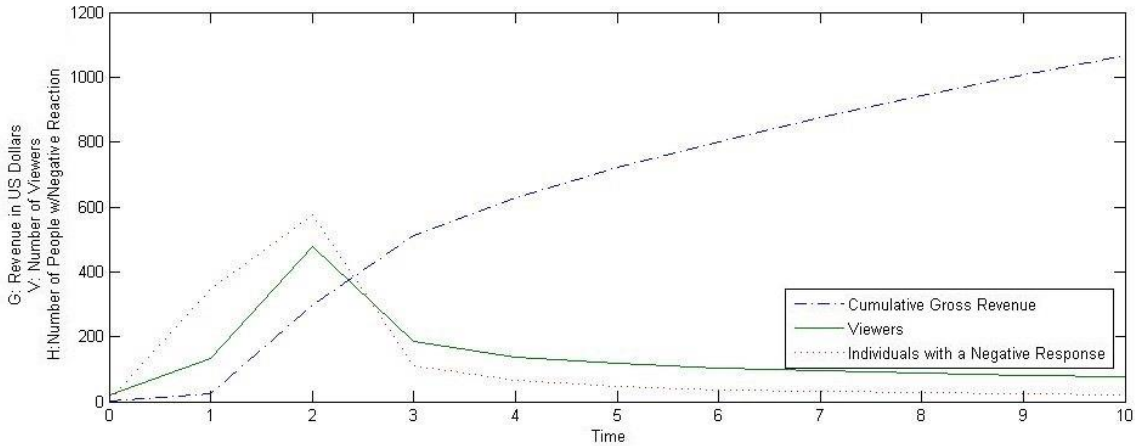
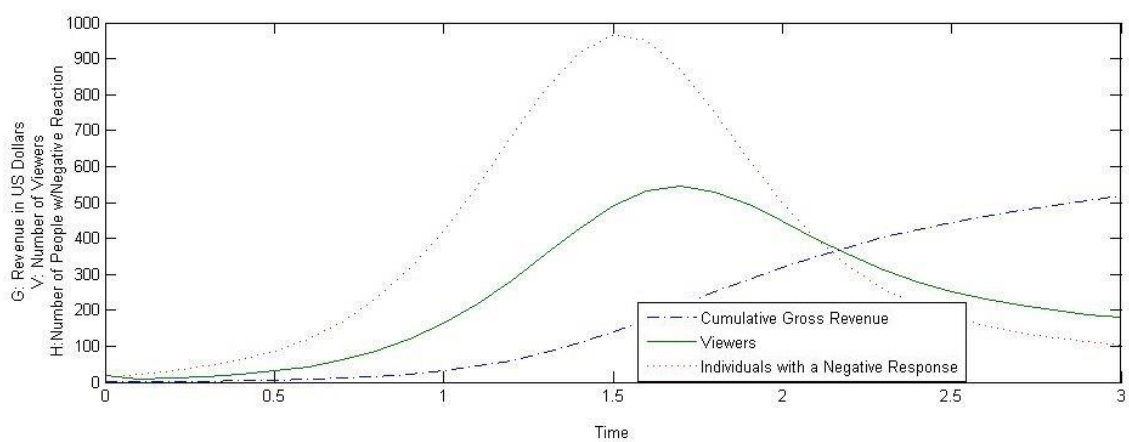


Figure 5: $P_4 = P_5 = P_6 = 10$, over $t \in (0, 3)$



Since increasing the values of the parameters by an order of magnitude resulted in increased instability and a reduction in the interval of stability, then decreasing the size of the parameters by an order of magnitude should result in improved stability and an increase of the interval of stability. We can see from Figure 5 that decreasing the parameters, so that $P_4 = P_5 = P_6 = .1$, results in the system being functionally stable over the interval $t \in (0, 100)$. Furthermore, if we graph the function over the interval $t \in (0, 500)$ we can see that the system remains functionally stable over the entire period (Figure 6).

Accordingly, while the system is generally unstable in theory, for sufficiently small values, it becomes stable in practice. Therefore, given that all parameters must be greater than zero it is implied that for $P_4, P_5, P_6 \in (0,1)$, the system is fundamentally stable. This is consistent with the existing use of statistical models as a predictive tool for analyzing behavior of network television, and thus a differential model is related to the statistical techniques used to analyze network television with respect to the construction of the parameters used in creating it [8].

Figure 6: $\boldsymbol{P_4 = P_5 = P_6 = .1}$**, over** $\boldsymbol{t \in (0, 100)}$

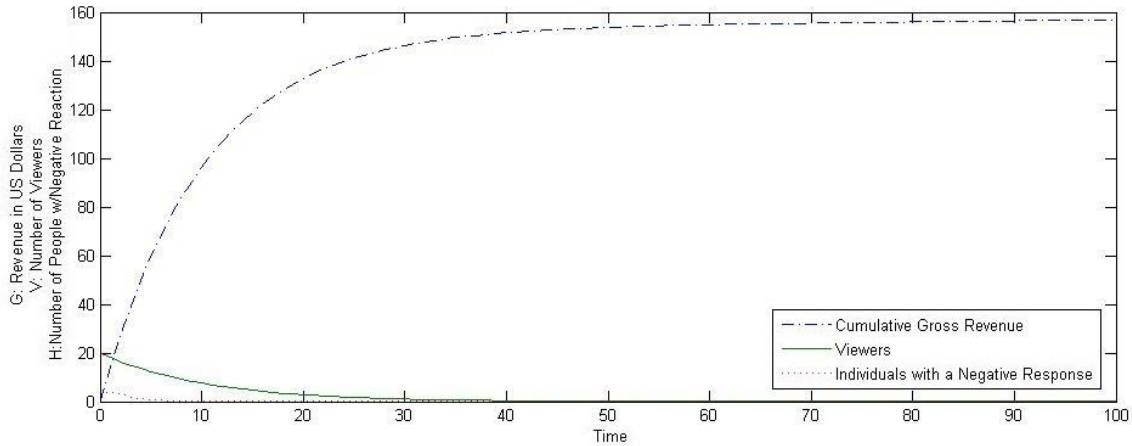
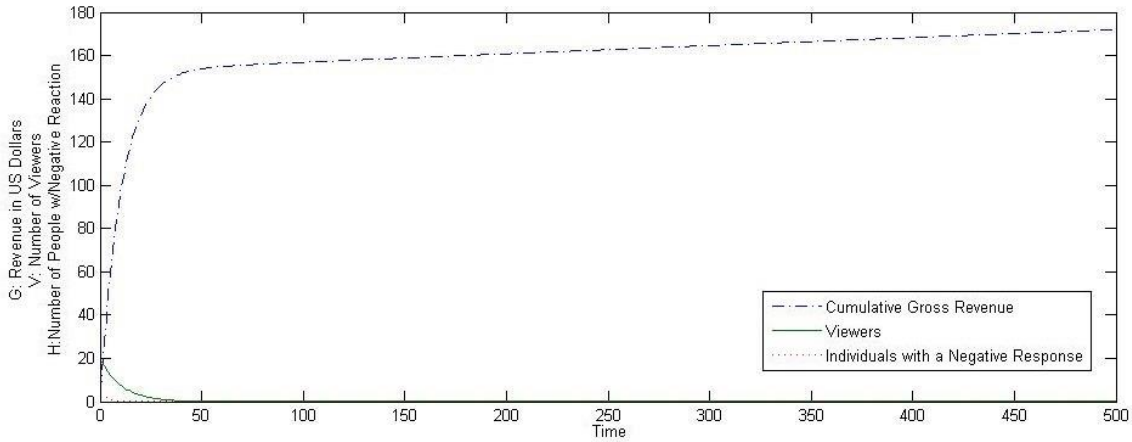


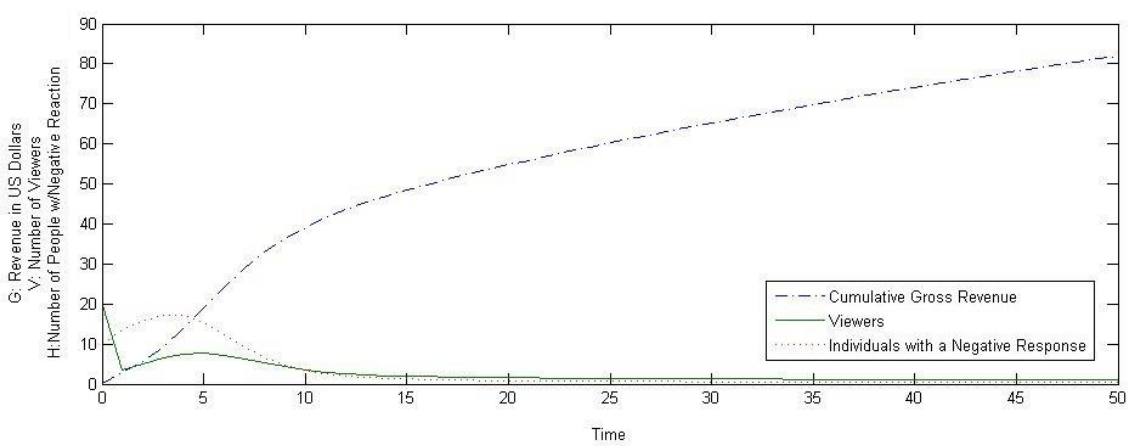
Figure 7: $\boldsymbol{P_4 = P_5 = P_6 = .1}$**, over** $\boldsymbol{t \in (0, 500)}$



Having analyzed the effects in changes of magnitude on the stability of our model, next we should consider how changes in the size of P_4, P_5, and P_6 relative to each other affect the system, while concentrating on circumstances that would result in a limited interval of stability. Not all alterations will have a substantial impact on the functions being modeled.

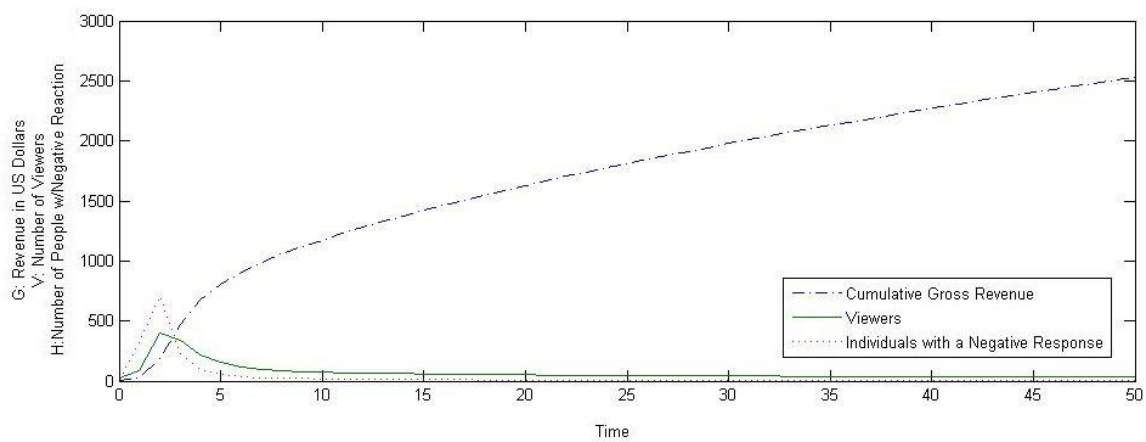
Increasing the size of P_4, while keeping both P_5, and P_6 smaller (Figure 7) results in both a rapid drop in the number of viewers, followed by a brief resurgence; and a bump in individuals with a negative reaction to the program. As those with a negative reaction exceeds the number of viewers for $t \in (1,10)$, this likely presents a case that network television programming directors would find undesirable.

Figure 8: $P_4 = 10$; $P_5 = P_6 = 1$, over $t \in (0, 50)$



Increasing the size of P_6, while keeping both P_4, and P_5 smaller, results in both a rapid spike in the number of individuals with a negative reaction to the program in the number of viewers, followed by an equally rapid drop; and a quick rise in the number of viewers, which then decays steadily (Figure 8). Those with a negative reaction exceeds the number of viewers for $t \in (1,4)$. This particular example indicates a situation that networks with a high level of tolerance for controversy may find acceptable. A temporary period of controversy attracts viewers and since the negative reaction is not sustained the network may well consider it an acceptable trade-off. Other variations in P_4, P_5, and P_6 result in similar outcomes that differ primarily in scale.

Figure 9: $\boldsymbol{P_6 = 10;\ P_4 = P_5 = 1}$**, over** $\boldsymbol{t \in (0, 50)}$



4.1.2 Variation of Initial Conditions

Having examined the effect of the parameters on the solution and stability of our model, we must also look at what effect, if any, the initial conditions have. Specifically for H_0 and V_0, and their relationship to each other, and what occurs when $\frac{H_0}{V_0} > P_3$. We will consider the functionally stable case where $P_4 = P_5 = P_6 = .1$ (Figure5); the extreme case of instability (Figure 3), where $P_4 = P_5 = P_6 = 10$; and also the milder case of instability (Figure 1), $P_4 = P_5 = P_6 = 1$.

For the case where the system is stable in practice, there is no significant difference in behavior for $\frac{H_0}{V_0} > P_3$ versus $\frac{H_0}{V_0} < P_3$, this emphasizes the near stability of the system when all parameters are elements of (0,1). Similarly, when the system is extremely unstable altering the initial conditions neither improves nor worsens the stability of the model.

When $P_4 = P_5 = P_6 = 1$, modifying the initial conditions has no effect on the interval of stability, but it does impact what the maximum percentage of negative reactions is relative to the number of viewers, as seen in Figures 9 and 10 the larger $\frac{H_0}{V_0}$ is the closer the maximum value of H is to intercepting V.

Figure 10: $\frac{H_0}{V_0} = .05$**, over** $t \in (0, 15)$

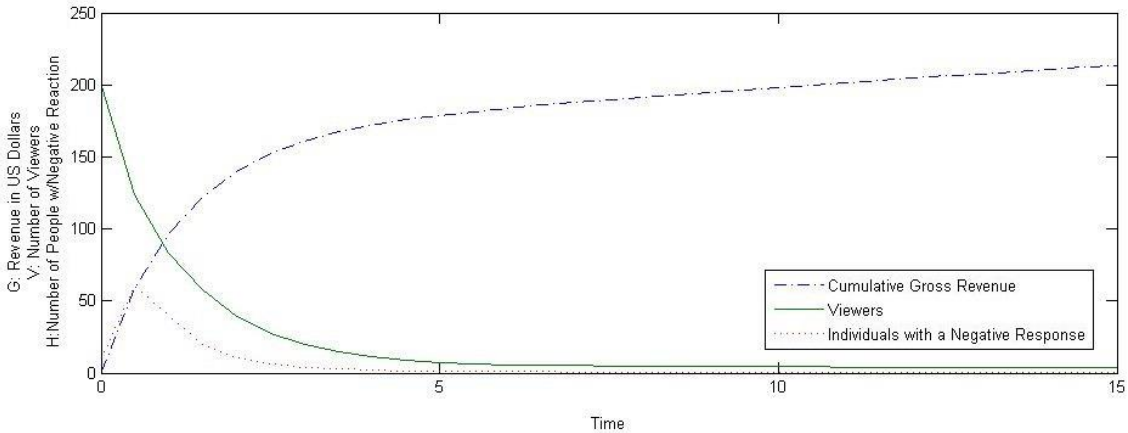
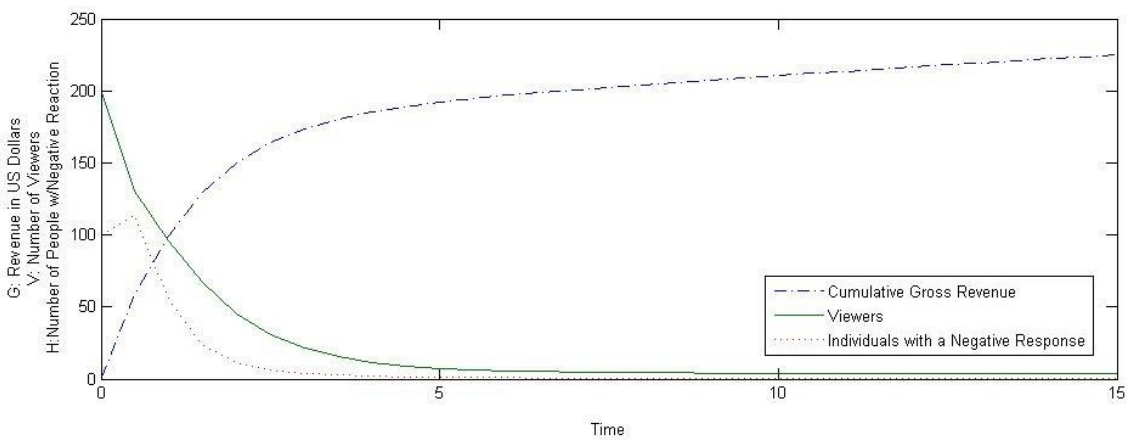


Figure 11: $\frac{H_0}{V_0} = .5$**, over** $t \in (0, 15)$



4.2 Eigenvalue Method

By the eigenvalue method we know that a system of differential equations is stable when the real part of all eigenvalues is less than zero, and the system may be asymptotically stable when the real part of all eigenvalues is less than or equal to zero. To find the eigenvalues for our system we must first find the characteristic equation for the system by and evaluating the Jacobian at the equilibrium point.

First we must identify any equilibrium points (Table 4) and find the Jacobian for the system.

Table 4: Equilibrium Points

G	$\frac{P_6}{P_7}$
V	$\left(\frac{P_5}{P_4}\right)\left(\frac{P_6}{P_7}\right)\left(\left(\frac{P_1}{P_2}\right)+P_3\right)^2$
H	$\left(\frac{P_5}{P_4}\right)\left(\frac{P_6}{.P_7}\right)\left(\left(\frac{P_1}{P_2}\right)+P_3\right)^3$

$$\begin{array}{cc} & \begin{array}{ccc} G & V & H \end{array} \\ \mathcal{J} = & \begin{bmatrix} 0 & P_1 + P_2P_3 & -P_2 \\ P_5\left(\frac{H}{V}\right)^2 & -P_4 - 2P_5\frac{GH^2}{V^3} & 2P_5\frac{GH}{V^2} \\ -P_7H & 0 & P_6 - P_7G \end{bmatrix} \begin{array}{c} G \\ V \\ H \end{array} \end{array} \quad (4.4)$$

Evaluating the Jacobian at the equilibrium point gives:

$$\mathcal{J} = \begin{bmatrix} 0 & P_1 + P_2P_3 & -P_2 \\ P_5\left(\frac{P_1}{P_2} + P_3\right)^2 & -3P_4 & 2\frac{P_2P_4}{P_1 + P_2P_3} \\ -\frac{P_5P_6}{P_4}\left(\frac{P_1}{P_2} + P_3\right)^2 & 0 & 0 \end{bmatrix} \quad (4.5)$$

For the purpose of analysis some values of P will be approximated where reasonable values may be inferred.

(1) $P_1 = \frac{cpm}{1000}\varphi$

The average cpm for network television in 2013 was $25. Advertising is typically sold in 15/30/60 second slots; for the sake of calculations we will assume an average time of 30 seconds per ad with approximately 15 minutes of ad time per one hour show.

$$P_1 = \frac{25 * 30}{1000} = 0.75$$

(2) $P_2 = P_1 n$; where n is the percentage of $(\frac{H}{V} - H_{\%})$

$$P_2 = .75 * .05 = .0375$$

(3) $P_3 = .2$; when H exceeds 20% of V it negatively affects G.

$P_3 \in [0,1]$

(4) $P_7 = -qk$

Where $q \in [0,1]$ q: representation of the perceived quality of the program, and $k \in [0,1]$ k: percentage of G reinvested into production

$P_7 \in [0,1]$, and $P_7 = -.70 * .05 = .035$

Plugging these values into 4.4, we get:

$$\mathcal{J} = \begin{bmatrix} 0 & .7575 & -.0375 \\ 408.04P_5 & -3P_4 & .099P_4 \\ -8242.408\left(\frac{P_5P_6}{P_4}\right) & 0 & 0 \end{bmatrix} \quad (4.6)$$

From this we get the characteristic equation:

$$-\lambda^3 - 3P_4\lambda^2 + \left(309.0903\frac{P_5P_6}{P_4} + 309.0903P_5\right)\lambda + (309.0903P_5P_6) = 0 \quad (4.7)$$

Dividing through by -1 to make the leading term positive gives:

$$\lambda^3 + 3P_4\lambda^2 - \left(309.0903\frac{P_5P_6}{P_4} + 309.0903P_5\right)\lambda - (309.0903P_5P_6) = 0 \quad (4.8)$$

Computation of the eigenvalues from a third order polynomial with three unknowns is computationally expensive. Accordingly, we will make use of a more efficient method for evaluating stability.

4.3 Routh-Hurwitz Stability Criterion

The stability of the system is dependent on the sign of the eigenvalue which is determined by the interactions between parameters. From the Routh-Hurwitz stability criterion, a system with the characteristic equation in the form of a third order polynomial: $n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0 = 0$ is stable if all coefficients are positive and $n_2 n_1 - n_3 n_0 > 0$ [11].

Accordingly, from equation 4.8, stability is contingent on satisfying the following inequalities

i. $3P_4 > 0$

ii. $-P_5(309.0903P_6 + 309.0903P_4) > 0$

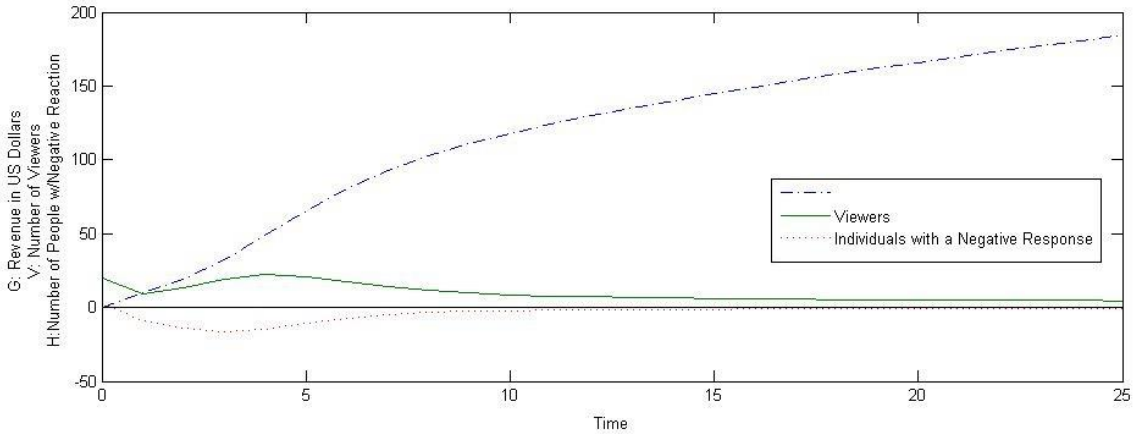
iii. $(309.0903P_5P_6) > 0$

iv. $-P_5(927.2709P_4 - 618.1806P_6) > 0$

These inequalities are satisfied when; $P_4 > 0$, $P_5 > 0$, and $P_6 < -1.5P_4 < 0$. Attempting to use a negative number for P_6 results in the number of people with a negative response to the program being given as a negative value (Figure 11). Given that $H(t)$ is a count of those who dislike the program its output cannot be less than zero, this implies that the system is unstable in general.

While the Routh-Hurwitz criteria proves that the system is generally unstable, our earlier graphs show that it is possible to get stable results over finite intervals of t, or results that are stable in practice for sufficiently small values of P_4, P_5, and P_6.

Figure 12: $P_4 < 0$



5 Optimization

A network may best be able to maximize the stability of the model and thus retain it as a helpful tool for analyzing the dynamics of network television by minimizing P_4, P_5, and P_6.

Table 5: Terms for Minimization

$\boldsymbol{P_4}$	$\left(\frac{\alpha_V}{E_{\%}(1+\epsilon)}\right)\left(\frac{\psi+C}{1+\gamma M}\right)$
$\boldsymbol{P_5}$	$\left(\frac{\alpha_V}{E_{\%}(1+\epsilon)}\right)\beta$
$\boldsymbol{P_6}$	$\left(\frac{\alpha_H}{1+\epsilon}\right)\left(\frac{1-\mathcal{S}}{1+\gamma M}\right)$

This can be accomplished by optimizing those parameters within the terms in table 5; ideally we should look to maximize/minimize all parameters contained in P_4, P_5, and P_6, however not all of the parameter in the system are within the networks sphere of control. Consequently, the most a network can achieve is optimization of those parameters which they have the ability to effect.

5.1 Minimization of P_4

To control the rate at which viewership declines, we must minimize P_4,the coefficient of V, to return to the parameters from the original form of the model:

$$P_4 = \left(\frac{\alpha_V}{E_{\%}(1+\epsilon)}\right)\left(\frac{\psi+C}{1+\gamma M}\right) \qquad (5.1)$$

Of the parameters comprising P_4, the network can only influence five. Of which 3 must be minimized, α_V, ψ, and C; and two should be maximized, γ and M. The decay

rate for viewership (α_V) can be minimized by producing programs with high production values, appropriate casting, and targeting popular genres for production. Once the network has created an appealing program that combines elements of what viewers want to watch, they must also schedule it for a timeslot with minimal competition (C) and minimize any disruptions (ψ) in airing the program. Finally, the television network needs to maximize γ by identifying and utilizing an effective marketing strategy, and allocate a marketing budget (M) sufficient to implement it.

5.2 Minimization of P_5

Having optimized the rate at which viewers are leaving the audience, we next want to control the growth of P_5. This term controls the rate at which viewers enter the audience; it is necessary to minimize P_5, as G is considerably larger than V, and it would allow growth to quickly outpace decay at an exponential rate if left unimpeded.

$$P_5 = \left(\frac{\alpha_V}{E_{\%}(1+\epsilon)}\right)\beta \qquad (5.2)$$

There is only one term, α_v, in P_5 that the network has any control over. All the other parameters are outside of the network's ability to influence. Accordingly, the only means the network has available to minimize P_5 is to minimize the decay rate for V.

5.3 Minimization of P_6

Finally, we want to control P_6 which is the rate at which people develop a negative reaction to the program. To do this we must consider those parameters in P_6, we are able to affect.

$$P_6 = \left(\frac{\alpha_H}{1+\epsilon}\right)\left(\frac{1-\mathcal{S}}{1+\gamma M}\right) \qquad (5.3)$$

P_6 contains four parameters which are within the networks' sphere of control, three of which need to be maximized, γ, M and $\mathcal{S}$; and one which needs to be minimized, α_H. The parameters γ and M may be maximized in the same manner as in P_4. While the growth rate for negative responses (α_H) should be minimized by producing programs with high production values, casting skilled actors with minimal negative public sentiment, and targeting popular and non-controversial genres, and managing negative publicity with regards to cast members. Once the network has created programming that combines elements of what viewers want to watch, they must also effectively manage potential negative publicity and controversies to maintain positive social factors $(\mathcal{S})$.

6 Conclusion

From the analysis of the model, we can conclude that the system is stable in practice over finite intervals of t, the length of the interval is dependent on the magnitude of the parameters, and in most cases that is adequate.

For analyzing a single season of a television show it is sufficient to have stability for the interval $t\epsilon(0,25)$ as most television programs have less than twenty-five episodes per season.

Cases with greater instability, and consequently small intervals of stability, may still be acceptable for short term use, such as evaluating a short term summer program, a miniseries, or a midseason replacement for a canceled series. These types of programs have seasons of shorter duration than a standard Fall/Spring series, and thus, a smaller interval may be acceptable in this instance.

Of particular importance, are those instances where $H > V$ for a large portion of the interval of stability. For most networks, the risks of carrying a show with such a strong negative perception, and the resulting loss of advertising support, far outweighs any potential increase in the number of viewers. Similarly, for cases where the initial conditions are such that $\frac{H_0}{V_0} > P_3$, the maximum value of negative reactions will have a higher peak and as a result negative reactions can exceed the number of viewers by a considerable margin.

While some elements cannot be controlled by the network, those that are under the network's influence can be altered as necessary to control the impact on the system.

The model could then be used to assist programming directors in correctly, and quickly, identifying fad versus sustainable programming and utilizing them appropriately as components of their programming schedules.

7 Limitations

There are some limitations immediately observable in this model. First ads are frequently sold in advance at prices which are set during sweeps [12], which occurs a few times during the year, for the sake of simplicity the model assumes that ad revenue is based on week to week viewership to avoid differing indices when evaluating the system. Also, the term P_G in equation (3.3) is treated as a constant when it is likely to be better represented as a function; this was done to streamline the model.

We also chose not to examine the effect of reruns on viewership and ad revenue; instead restricting the model to first run episodes. This choice ignores what is likely a fairly large amount of potential revenue in doing so, as there are typically 23 episodes in a season and 52 weeks in a year. Granted, some networks choose to air shorter, 13 episode series, in lieu of airing reruns during at least some portion of the year. Most episodes are repeated at least once during their timeslot after the original broadcast or are rebroadcast in another time slot. These airings then can, potentially, represent a significant source of additional ad revenue.

Perhaps the most important limitation present is the fact that a considerable amount of the data needed to run the model is not currently collected, this means that the model is necessarily a conceptual model and cannot be tested against any data to determine its accuracy.

8 Future Considerations

The most logical course for future action would be to conduct studies to collect data on those parameters which are not currently measured, and to discover the actual behavior of the negative response to a program over a season, or several seasons, to compare the model against and make necessary modifications to improve accuracy.

In constructing our model, the cost of viewership is considered only in monetary terms. There may be other non-monetary costs associated with watching multiple television programs, which should be factored into any subsequent work.

Our model also assumes that a percentage of gross revenue is reinvested into the program from the first penny. It is more likely that after the initial production and marketing expenditures, networks wait to reinvest in a program until after the gross revenue has recouped the initial investment plus some cushion to cover losses from poorly preforming shows. Some means of accounting for this should be included in future refinements of the model. Furthermore, the percentage of the gross reinvested into controlling H, (P_G), is treated as a parameter, when it is far more likely that this is a function with respect to time. In the future we may wish to model this by adding a fourth equation to the model.

Alternatively, we may wish to examine the effects of positive response to a program as a fourth equation, in addition to the effects of negative reactions. This would be expected to affect all three of the equations in our model, and may provide a replacement for $\left(\frac{H}{V} - H_{\%}\right)$ in $\frac{dG}{dt}$ by examining the difference between positive and negative opinions of the program.

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Appendix: Computer Code

```
function [ drv ] = TV5( t,y )
P=[.75, ...
   3.75*10^(-2),...
   .2, ...
   10^(0), ...
   10^(0), ...
   10^(0), ...
   3.5*10^(-2)];

G=y(1);
V=y(2);
H=y(3);

drv=zeros(3,1);

drv(1)=P(1)*V-P(2)*(H/V-P(3))*V;
drv(2)=-P(4)*V+P(5)*G*(H/V)^2;
drv(3)=P(6)*V-P(7)*G*H;

end
```

```
function graph(X1, YMatrix1)
%CREATEFIGURE(X1,YMATRIX1)
%  X1:  vector of x data
%  YMATRIX1:  matrix of y data

%  Auto-generated by MATLAB on 18-Jul-2015 14:37:16

% Create figure
figure1 = figure;

% Create axes
axes1 = axes('Parent',figure1);
box(axes1,'on');
hold(axes1,'all');

% Create multiple lines using matrix input to plot
plot1 = plot(X1,YMatrix1,'Parent',axes1);
set(plot1(1),'LineStyle','-.','DisplayName','Cumulative Gross Revenue');
set(plot1(2),'DisplayName','Viewers');
set(plot1(3),'LineStyle',':',...
    'DisplayName','Individuals with a Negative Response');

% Create legend
legend1 = legend(axes1,'show');
set(legend1,...
    'Position',[0.636209029066173 0.176445578231292 0.250463821892393
0.154761904761905]);
```