EXISTENCE OF A UNIQUE SOLUTION TO A SYSTEM OF EQUATIONS MODELING COMPRESSIBLE FLUID FLOW WITH CAPILLARY STRESS EFFECTS

A Thesis

by

LANE COSPER

BS, Texas A&M University - Corpus Christi, 2016

Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER of SCIENCE

in

MATHEMATICS

Texas A&M University-Corpus Christi
Corpus Christi, Texas

May 2018

EXISTENCE OF A UNIQUE SOLUTION TO A SYSTEM OF EQUATIONS MODELING COMPRESSIBLE FLUID FLOW WITH CAPILLARY STRESS EFFECTS

A Thesis

by

LANE COSPER

This thesis meets the standards for scope and quality of
Texas A&M University-Corpus Christi and is hereby approved.

Diane Denny, PhD
Chair

Devanayagam Palaniappan, PhD
Committee Member

Alexey Sadovski, PhD
Committee Member

May 2018

ABSTRACT

The purpose of this thesis is to prove the existence of a unique solution to a system of partial differential equations which models the flow of a compressible barotropic fluid under periodic boundary conditions. The equations come from modifying the compressible Navier-Stokes equations. The proof utilizes the method of successive approximations. We will define an iteration scheme based on solving a linearized version of the equations. Then convergence of the sequence of approximate solutions to a unique solution of the nonlinear system will be proven. The main new result of this thesis is that the density data is at a given point in the spatial domain over a time interval instead of an initial density over the entire spatial domain. Further applications of the mathematical model are fluid flow problems where the data such as concentration of a solute or temperature of the fluid is known at a given point. Future research could use boundary conditions which are not periodic.

ACKNOWLEDGEMENTS

I would like to take this opportunity to thank several people who were played essential roles in the process of my thesis. First, the chair of my thesis committee, Dr. Diane Denny. Thank you for guiding and assisting me. This process would not have been as smooth as it was without you. Next, I would like to thank the other two members of my thesis committee, Dr. Devanayagam Palaniappan and Dr. Alexey Sadovski. Thank you for your time, effort, and consideration. I would also like to acknowledge the rest of the faculty and staff in the Mathematics and Statistics Department at Texas A& M University Corpus Christi. Specifically, I would like to acknowledge Dr. Beate Zimmer for acting as an adviser in all matters regarding the thesis process. Lastly, I would like state my sincerest appreciation for my friends and family. Thank you for supporting me throughout my Bachelor's and Master's degree.

TABLE OF CONTENTS

CONTENTS PAGE

CHAPTER I: INTRODUCTION

The purpose of this thesis is to prove the existence of a unique solution to a system of partial differential equations which models a compressible barotropic fluid. Since the fluid is compressible, any change in pressure results in a change to the volume. A barotropic fluid is one who's pressure is only dependent upon density. The equations in this thesis come from modifying the compressible Navier-Stokes equations. In our equations we have included a capillary stress term. The modified conservation of mass equation uses an approximation to the divergence of velocity. Our system's dependent variables are the velocity and density of the fluid.

The data for our system consists of an initial velocity at time $t = 0$ for all $x \in \Omega$, where $\Omega = \mathbb{T}^2$, the two-dimensional torus, is the spatial domain, the density at a single point $x_0 \in \mathbb{T}^2$ for $0 \leq t \leq T$, and the divergence of velocity for $0 \leq t \leq T$ and for all $x \in \mathbb{T}^2$. This thesis will prove the existence of a unique solution to the system of modified Navier-Stokes equations under periodic boundary conditions for the time interval $0 \leq t \leq T$. Using density data at a given point in the spatial domain instead of using initial density is the main new result of this thesis.

The proof in this thesis will utilize the method of successive approximations. We will define an iteration scheme based on solving a linearized version of the equations. Then convergence of the sequence of approximate solutions to a unique solution of the nonlinear system will be proven. The main theorem to be proven appears in Chapter 4. The proof of the existence of a solution to the linearized equations appears in Appendix A. The convergence of the sequence of solutions is proven in Chapter 5. Appendix B contains lemmas supporting the proof. We will begin by reviewing the literature in Chapter 2 and then present the model's equations in Chapter 3.

CHAPTER II: REVIEW OF THE LITERATURE

The system of equations to be studied is a modified version of the Navier-Stokes equations for a compressible barotropic fluid, which consist of partial differential equations for the balance of linear momentum and for the conservation of mass. The existence of a solution to the Navier-Stokes equations has been studied by many researchers under the condition that initial data for the density be given. For example, Bresch, Desjardins and Gerard-Varet [2], Choe and Kim [3], Desjardins [10], Hoff [14, 15], and Mellet and Vasseur [16] proved the existence of a solution to a system of equations modeling the flow of a compressible, barotropic fluid when the initial value of the density is given. None of the work done by these researchers uses the condition that the value of the density is specified at a point $\mathbf{x}_0$ in the domain, and that the initial value of the density is not given.

The methodology used in my thesis to prove the existence of a unique solution to this system of equations is similar to the methodology appearing in Embid [11, 12], which he uses to prove the existence of a solution to equations modeling zero-Mach number combustion by proving convergence of a sequence of approximate solutions to an iteration scheme based on a linearized version of the system of equations. In other related work Denny in [5, 6, 7] proves the existence of a solution to a quasilinear elliptic equation with data at a spatial point. In [8], Denny proves the existence of a solution to a system of equations modeling the flow of a nearly incompressible, inviscid fluid with a capillary stress term, and with density data at a point in the spatial domain. Song [17] uses an iteration scheme to prove the existence and uniqueness of a solution to a system of equations which model compressible fluid flow. Song's equations do not include a term for convection or capillary stress effects. The system of equations being studied in my thesis is a system of equations which is different from the equations studied by Denny, Embid, and by Song.

CHAPTER III: MODEL'S EQUATIONS

3.1 The Mathematical Model

The Navier-Stokes equations for a compressible barotropic fluid are

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p = \nu \Delta \mathbf{v} + \gamma \nabla(\nabla \cdot \mathbf{v}) \tag{3.1}$$

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\rho_t - \frac{1}{\rho}(\mathbf{v} \cdot \nabla \rho) \tag{3.2}$$

$$p = \hat{p}(\rho) \tag{3.3}$$

where ρ is the density, $\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$ is the material derivative, $\mathbf{v}$ is the velocity, p is the pressure, ν and γ are coefficients of viscosity, which are positive constants.

The Navier-Stokes equations are modified to include capillary stress effects by adding a term $c\rho\nabla\Delta\rho$, which produces the equations:

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p = \nu \Delta \mathbf{v} + \gamma \nabla(\nabla \cdot \mathbf{v}) + c\rho\nabla\Delta\rho \tag{3.4}$$

where the capillary stress coefficient c is a positive constant.

We will be studying a modified version of equations (3.4), (3.2), and (3.3). The first notable modification we will make is to assume that the conservation of mass equation (3.2) can be approximated by:

$$\nabla \cdot \mathbf{v} = f \tag{3.5}$$

where f is a given function such that $\int_\Omega f d\mathbf{x} = 0$.

Next, from the equation of state (3.3) it follows that $\nabla p = p^{'}(\rho)\nabla\rho$. Applying this to our equation (3.4) we obtain:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{p^{'}(\rho)}{\rho}\nabla\rho = \frac{\nu}{\rho}\Delta \mathbf{v} + \frac{\gamma}{\rho}\nabla(\nabla \cdot \mathbf{v}) + c\nabla\Delta\rho \tag{3.6}$$

To have a well-posed problem boundary conditions should be given. We apply periodic boundary conditions on a two-dimensional spatial domain $\mathbb{T}^2 = [0,2\pi]x[0,2\pi]$ so that:

$$\mathbf{v}(0,y,t) = \mathbf{v}(2\pi,y,t) \tag{3.7}$$

$$\mathbf{v}(x,0,t) = \mathbf{v}(x,2\pi,t) \tag{3.8}$$

for $x \in [0,2\pi]$ and $y \in [0,2\pi]$.

The periodic boundary conditions are also applied to $\rho(\mathbf{x},t)$. The given data for the initial velocity $\mathbf{v}(\mathbf{x},0)$ for all $\mathbf{x} \in \Omega$ and for the density $\rho(\mathbf{x}_0,t)$ at a spatial point $\mathbf{x}_0 \in \Omega$ for $0 \leq t < T$ is:

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \tag{3.9}$$

$$\rho(\mathbf{x}_0,t) = g(t) \tag{3.10}$$

where $g(t)$ is a given positive smooth function, and $\mathbf{v}_0(x)$ is a given smooth periodic function. Our model's equations are given by (3.5)-(3.10).

3.2 Non-Dimensionalization

Now we will non-dimensionalize the equations (3.6) and (3.5). To do so, we define:

- $\mathbf{x}^* = \frac{\mathbf{x}}{L}$
- $\mathbf{v}^* = \frac{\mathbf{v}}{|\mathbf{v}_m|}$
- $\tau = \frac{t|\mathbf{v}_m|}{L}$
- $p^* = \frac{p}{|p_m|}$
- $\nabla^* = L\nabla$
- $\Delta^* = L^2\Delta$
- $\rho^* = \frac{\rho}{\rho_m}$
- $f^* = \frac{Lf}{|\mathbf{v}_m|}$

- $Re = \frac{\rho_m L|\mathbf{v}_m|}{\nu}$
- $\lambda = \frac{|p_m|}{\rho_m|\mathbf{v}_m|^2}$
- $c^* = \frac{\rho_m c}{L^2|\mathbf{v}_m|^2}$

where $|\mathbf{v}_m|$, L, $|p_m|$, and ρ_m is the average velocity, length, pressure, and density respectively, and Re is the Reynold's number. Now applying these to equations (3.5) and (3.6) and simplifying yields:

$$\frac{|\mathbf{v}_m|^2}{L}\frac{\partial \mathbf{v}^*}{\partial \tau} + \frac{|\mathbf{v}_m|^2}{L}\mathbf{v}^* \cdot \nabla^* \mathbf{v}^* + \frac{|p_m|}{\rho_m L}\frac{dp^*}{d\rho^*}(\rho_m \rho^*)\frac{\nabla^* \rho^*}{\rho^*}$$
$$= \frac{|\mathbf{v}_m|}{\rho_m L^2}\frac{\nu}{\rho^*}\Delta^* \mathbf{v}^* + \frac{|\mathbf{v}_m|}{\rho_m L^2}\frac{\gamma}{\rho^*}\nabla^*(\nabla^* \cdot \mathbf{v}^*) + \frac{\rho_m}{L^3}c\nabla^*\Delta^*\rho^* \tag{3.11}$$
$$\frac{|\mathbf{v}_m|}{L}\nabla^* \cdot (\mathbf{v}^*) = \frac{|\mathbf{v}_m|}{L}f^* \tag{3.12}$$

By plugging in the Re and λ we obtain the standard equations:

$$\mathbf{v}^*_\tau + \mathbf{v}^* \cdot \nabla^* \mathbf{v}^* + \frac{\lambda}{\rho^*}\frac{dp^*}{d\rho^*}(\rho_m \rho^*)\nabla^* \rho^* = \frac{1}{Re\rho^*}(\Delta^* \mathbf{v}^* + \frac{\gamma}{\nu}\nabla^*\nabla^* \cdot \mathbf{v}^*) + c^*\nabla^*\Delta^*\rho^* \tag{3.13}$$

$$\nabla^* \cdot \mathbf{v}^* = f^* \tag{3.14}$$

Equations (3.13) and (3.14) represent the non-dimensionalized equations. For notational convenience we will write equations (3.13) and (3.14) as follows:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\lambda}{\rho}p'(\rho)\nabla\rho = \frac{1}{Re\rho}(\Delta \mathbf{v} + \frac{\gamma}{\nu}\nabla\nabla \cdot \mathbf{v}) + c\nabla\Delta\rho \tag{3.15}$$

$$\nabla \cdot \mathbf{v} = f \tag{3.16}$$

The purpose of this thesis is to prove the existence of a unique solution to (3.15)-(3.16) under periodic boundary conditions with initial velocity data $\mathbf{v}_0(\mathbf{x})$ and with density data $\rho(\mathbf{x}_0, t) = \rho_0(t)$. The format of the proof follows one used by Embid [11] to prove the existence of a solution to equations for zero Mach number combustion. The format of the proof is separated into three steps. First, we prove the boundedness of the approximating sequence of solutions in a high Sobolev norm. Second, we prove contraction of the sequence in a low Sobolev norm. Finally, we prove convergence of the sequence to a unique solution of the nonlinear equations.

CHAPTER IV: MAIN THEOREM

The main theorem is as follows:

Main Theorem:

Let the spatial domain be $\Omega = \mathbb{T}^2$, the two-dimensional torus. Let p be a given smooth positive function of ρ and let $\frac{dp}{d\rho}$ be a positive function. Let ρ_0 be a given positive smooth function of t, and let $\mathbf{x}_0$ be a given point in $\mathbb{T}^2$. Let f be a smooth function of t and x such that $\int_\Omega f\,d\mathbf{x} = 0$. Then for a time interval $0 \le t \le T$, equations (3.15) and (3.16), subject to the condition that $\rho(\mathbf{x}_0, t) = \rho_0(t)$ and the initial condition $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(x) \in H^s(\mathbb{T}^2)$ where $s > \frac{N}{2} + 4$ and $N = 2$ have a unique solution ρ and $\mathbf{v}$ where ρ is a positive function, provided that:

$$T \le \delta$$
$$\frac{1}{Re^2} \le \delta^2$$
$$\frac{1}{\lambda^2} \le \delta^2$$

where $0 < \delta < 1$, and δ is sufficiently small. The regularity of the solution is

$$\mathbf{v} \in C([0,T], C^4(\mathbb{T}^2)) \cap L^\infty([0,T], H^s(\mathbb{T}^2) \cap L^2([0,T], H^{s+1}(\mathbb{T}^2))$$
$$\rho \in C([0,T], C^5(\mathbb{T}^2)) \cap L^\infty([0,T], H^{s+1}(\mathbb{T}^2))$$
$$\frac{\partial \mathbf{v}}{\partial t} \in C([0,T], C^2(\mathbb{T}^2)) \cap L^\infty([0,T], H^{s-2}(\mathbb{T}^2))$$

Proof:

First, we apply the Helmholtz Decomposition to equations (3.15) and (3.16). Helmholtz Decomposition is a way of uniquely writing a vector field as the sum of a solenoidal vector and a gradient vector. A solenoidal vector is a vector $\mathbf{w}$ such that $\nabla \cdot \mathbf{w} = 0$. To perform the Helmholtz Decomposition we must let $\mathbf{v} = \mathbf{w} + \nabla\phi$ where $\nabla \cdot \mathbf{w} = 0$. Applying this to equations (3.15) and (3.16) we get:

$$(\mathbf{w}+\nabla\phi)_t+(\mathbf{w}+\nabla\phi)\cdot(\mathbf{w}+\nabla\phi)+\lambda\frac{p^{'}(\rho)}{\rho}\nabla\rho$$
$$=\frac{1}{Re\rho}(\Delta(\mathbf{w}+\nabla\phi)+\frac{\gamma}{\nu}\nabla(\nabla\cdot(\mathbf{w}+\nabla\phi)))+c\nabla\Delta\rho \tag{4.1}$$

$$\nabla\cdot(\mathbf{w}+\nabla\phi)=f \tag{4.2}$$

Further simplification gives:

$$\mathbf{w}_t+\nabla\phi_t+(\mathbf{w}+\nabla\phi)\cdot\nabla(\mathbf{w}+\nabla\phi)+\frac{\lambda p^{'}(\rho)\nabla\rho}{\rho}=\frac{1}{Re\rho}(\Delta\mathbf{w}+(1+\frac{\gamma}{\nu})\nabla\Delta\phi)+c\nabla\Delta\rho \tag{4.3}$$

$$\nabla\cdot\mathbf{w}=0 \tag{4.4}$$

$$\Delta\phi=f \tag{4.5}$$

We will construct the solution using the following iteration scheme:

$$\mathbf{w}_t^{k+1}+\nabla\phi_t+(\mathbf{w}^k+\nabla\phi)\cdot\nabla(\mathbf{w}^{k+1}+\nabla\phi)+\lambda\frac{p^{'}(\rho^k)\nabla\rho^{k+1}}{\rho^k}$$
$$=\frac{1}{Re\rho^k}(\Delta\mathbf{w}^{k+1}+(1+\frac{\gamma}{\nu})\nabla\Delta\phi)+c\nabla\Delta\rho^{k+1} \tag{4.6}$$

$$\nabla\cdot\mathbf{w}^{k+1}=0 \tag{4.7}$$

$$\Delta\phi=f \tag{4.8}$$

$$\rho^{k+1}(\mathbf{x}_0,t)=\rho_0(t) \tag{4.9}$$

$$\mathbf{w}^{k+1}(\mathbf{x},0)=\mathbf{w}_0(\mathbf{x}) \tag{4.10}$$

From the initial velocity condition we let the initial solenodial velocity iterate $\mathbf{w}^0=\mathbf{w}_0(x)$, and from the density condition $\rho(\mathbf{x}_0,t)=\rho_0(t)$ at a spatial point $\mathbf{x}_0$ we let the initial density iterate be $\rho^0=\rho_0(t)$.

Equations (4.6)-(4.8) linearizes the system given by (4.3)-(4.5). The next step to proving the existence of a unique solution to the nonlinear system is to prove the existence of a solution to the linear system for each fixed k. The proof of the existence of a solution to the linear equations appears in Appendix A. Now, we proceed to prove convergence of the sequence of approximate solutions to a solution of the nonlinear system (4.3)-(4.5).

CHAPTER V: Proof of the Theorem

5.1 High Sobolev Norm Bounds

To prove the convergence of the sequence of solutions to the system of equations (4.6)-(4.8), we start by determining a high norm Sobolev space bound. We proceed with the proof by mathematical induction. For this we assume:

$$\begin{aligned}\|\mathbf{w}^k\|_s^2 + \frac{c_2}{Re}\int_0^t \|\nabla \mathbf{w}^k\|_s^2 d\tau \leq L_1^2 \\ \|\nabla\rho^k\|_{s-1}^2 \leq \delta^2 L_2^2 \\ \|\Delta\rho^k\|_{s-1}^2 \leq \frac{1}{c}L_2^2 \\ \|\rho^k\|_{s+1}^2 \leq L_3^2 \\ |\rho^k - \rho_0|_{L^\infty,T} < R\end{aligned}$$

where $\frac{p'(\rho^k)}{\rho^k} > c_1 > 0$ and $\frac{1}{\rho^k} > c_2 > 0$, and $\rho_0 - R > 0$ for $|\rho^k - \rho_0|_{L^\infty,T} < R$. We will prove that

$$\begin{aligned}\|\mathbf{w}^{k+1}\|_s^2 + \frac{c_2}{Re}\int_0^t \|\nabla \mathbf{w}^{k+1}\|_s^2 d\tau \leq L_1^2 \\ \|\nabla\rho^{k+1}\|_{s-1}^2 \leq \delta^2 L_2^2 \\ \|\Delta\rho^{k+1}\|_{s-1}^2 \leq \frac{1}{c}L_2^2 \\ \|\rho^{k+1}\|_{s+1}^2 \leq L_3^2 \\ |\rho^{k+1} - \rho_0|_{L^\infty,T} < R\end{aligned}$$

where $\mathbf{w}^{k+1}$ is the solution to

$$\begin{aligned}\mathbf{w}_t^{k+1} + \nabla\phi_t + (\mathbf{w}^k + \nabla\phi)\cdot\nabla(\mathbf{w}^{k+1} + \nabla\phi) + \lambda\frac{p'(\rho^k)}{\rho^k}\nabla\rho^{k+1} \\ = \frac{1}{Re\rho^k}(\Delta\mathbf{w}^{k+1} + (1+\frac{\gamma}{\nu})\nabla\Delta\phi) + c\nabla\Delta\rho^{k+1} \qquad (5.1)\end{aligned}$$

Applying Lemma B.3 to equation (5.1) with

$$
\begin{aligned}
a(\rho^k) &= \frac{p'(\rho^k)}{\rho^k} \\
b(\rho^k) &= \frac{1}{\rho^k} \\
\mathbf{v}^k &= \mathbf{w}^k + \nabla\phi \\
\mathbf{F} &= \frac{1}{Re\rho^k}(1+\frac{\gamma}{\nu})\nabla\Delta\phi - \nabla\phi_t - \mathbf{v}^k\cdot\nabla\nabla\phi
\end{aligned}
$$

and using Lemma B.1 gives us

$$
\begin{aligned}
\|\mathbf{w}^{k+1}\|_s^2 + \frac{c_2}{Re}\int_0^t \|D\mathbf{w}^{k+1}\|_s^2 d\tau &\leq C[\|\mathbf{w}_0\|_s^2 + \int_0^T \|\mathbf{F}\|_s^2 d\tau] \\
&\leq C[\|\mathbf{w}_0\|_s^2 + \int_0^t \|\frac{1}{Re\rho^k}(1+\frac{\gamma}{\nu})\nabla\Delta\phi - \nabla\phi_t - \mathbf{v}^k\cdot\nabla\nabla\phi\|_s^2 d\tau] \\
&\leq C\|\mathbf{w}_0\|_s^2 + C\int_0^t (\|\frac{1}{Re\rho^k}\|_s^2\|\nabla\Delta\phi\|_s^2 + \|\nabla\phi_t\|_s^2 \\
&\qquad + \|\mathbf{w}^k + \nabla\phi\|_s^2\|\nabla\phi\|_{s+1}^2)d\tau \\
&\leq C\|\mathbf{w}_0\|_s^2 + C\int_0^t (\frac{C_1}{Re^2}\|\nabla\Delta\phi\|_s^2 + \|\nabla\phi_t\|_s^2 \\
&\qquad + \|\mathbf{w}^k\|_s^2\|\nabla\phi\|_{s+1}^2 + \|\nabla\phi\|_{s+1}^4)d\tau \\
&\leq C\|\mathbf{w}_0\|_s^2 + C\int_0^t (\frac{C_1}{Re^2}\|\nabla f\|_s^2 + \|f_t\|_{s-1}^2 \\
&\qquad + L_1^2\|f\|_s^2 + \|f\|_s^4)d\tau \\
&\leq C\|\mathbf{w}_0\|_s^2 + CT[\frac{C_1}{Re^2}\|f\|_{s+1,T}^2 + \|f_t\|_{s-1,T}^2 \\
&\qquad + L_1^2\|f\|_{s,T}^2 + \|f\|_{s,T}^4] \\
&\leq C\|\mathbf{w}_0\|_s^2 + C_2 \\
&= L_1^2 \qquad (5.2)
\end{aligned}
$$

where $TL_1^2 \leq 1$, C depends on s, C_1 depends on s, ρ_0, R, c_1, and $\frac{C_1}{Re^2} \leq 1$, and C_2 depends on s, $\|f_t\|_{s-1,T}$, $\|f\|_{s+1,T}$ and where L_1 depends on s, $\|f_t\|_{s-1,T}$, $\|f\|_{s+1,T}$, and $\|\mathbf{w}_0\|_s^2$.

Next we take the divergence of equation (5.1) to obtain:

$$\begin{aligned} f_t + \nabla\cdot(\mathbf{v}^k\cdot\nabla(\mathbf{w}^{k+1}+\nabla\phi)) + \nabla\cdot(\lambda\frac{p^{'}(\rho^k)}{\rho^k}\nabla\rho^{k+1}) \\ = \nabla\cdot(\frac{\Delta\mathbf{w}^{k+1}}{Re\rho^k}) + \nabla\cdot(\frac{(1+\frac{\gamma}{\nu})\nabla f}{Re\rho^k}) + c\Delta^2\rho^{k+1} \end{aligned}$$

Then using Lemma B.1 and B.2 we obtain

$$\begin{aligned} c\|\Delta\rho^{k+1}\|_{s-1}^2 + \lambda c_1\|\nabla\rho^{k+1}\|_{s-1}^2 &\le \frac{C}{\lambda c_1}[\|D\frac{1}{Re\rho^k}\|_{s-1}^2\|\Delta\mathbf{w}^{k+1}\|_{s-2}^2 + \|\mathbf{v}^k\cdot\nabla\mathbf{w}^{k+1}\|_{s-1}^2 + \|\mathbf{F}\|_{s-1}^2] \\ &\le \frac{C}{\lambda c_1}[\|D\frac{1}{Re\rho^k}\|_{s-1}^2\|\mathbf{w}^{k+1}\|_s^2 + C\|\mathbf{v}^k\|_{s-1}^2\|\mathbf{w}^{k+1}\|_s^2 + \|\mathbf{F}\|_{s-1}^2] \\ &\le \frac{C}{\lambda c_1}\frac{1}{Re^2}\|D(\frac{1}{\rho^k})\|_{s-1}^2\|\mathbf{w}^{k+1}\|_s^2 \\ &\quad + \frac{C}{\lambda c_1}[(\|\mathbf{w}^k\|_{s-1} + \|\nabla\phi\|_{s-1})^2\|\mathbf{w}^{k+1}\|_s^2 + \|\mathbf{F}\|_{s-1}^2] \\ &\le \frac{C}{\lambda c_1}\frac{C_3}{Re^2}\|\nabla\rho^k\|_{s-1}^2\|\mathbf{w}^{k+1}\|_s^2 \\ &\quad + \frac{C}{\lambda c_1}[(L_1^2 + \|f\|_{s-2}^2)\|\mathbf{w}^{k+1}\|_s^2 + \|\mathbf{F}\|_{s-1}^2] \end{aligned}$$

where C depends on s and C_3 depends on s, ρ_0, R. We can now plug in the estimate from (5.2) to get

$$\begin{aligned} \|\Delta\rho^{k+1}\|_{s-1}^2 &\le \frac{C}{\lambda c_1 c}[(\frac{C_3}{Re^2}\delta^2L_2^2 + L_1^2 + \|f\|_{s-2,T}^2)L_1^2 + \frac{C_1}{Re^2}\|f\|_{s,T}^2 \\ &\quad + \|f_t\|_{s-2,T}^2 + L_1^2\|f\|_{s-1,T}^2 + \|f\|_{s-1,T}^4] \\ &\le \frac{1}{c}L_2^2 \end{aligned}$$

$$\begin{aligned} \|\nabla\rho^{k+1}\|_{s-1}^2 &\le \frac{C}{\lambda^2c_1^2}[(\frac{C_3}{Re^2}\delta^2L_2^2 + L_1^2 + \|f\|_{s-2,T}^2)L_1^2 + \frac{C_1}{Re^2}\|f\|_{s,T}^2 \\ &\quad + \|f_t\|_{s-2,T}^2 + L_1^2\|f\|_{s-1,T}^2 + \|f\|_{s-1,T}^4] \\ &\le \delta^2L_2^2 \end{aligned}$$

and by Lemma A.1

$$\|\nabla\rho^{k+1}\|_s^2 \le C(c\|\Delta\rho^{k+1}\|_{s-1}^2 + \lambda c_1\|\nabla\rho^{k+1}\|_{s-1}^2) \le CL_2^2 + C\lambda c_1\delta^2L_2^2$$

where $\frac{1}{\lambda} \le \delta$, $\frac{1}{Re^2} \le \delta^2$, and δ^2 is sufficiently small so that $\frac{\delta^2 L_2^2}{Re^2} \le 1$. Here L_2 depends on c_1, s, $\|f\|_{s+1,T}$, $\|f_t\|_{s-1,T}$, ρ_0, R and $\|\mathbf{w}_0\|_s$.

Since $\rho^{k+1}(\mathbf{x}_0,t) = \rho_0(t)$ at a single point and $\rho^0(\mathbf{x},t) = \rho_0(t)$, then by Lemma B.1

$$\begin{aligned}\|\rho^{k+1}\|_0^2 &\le C\|\rho^0\|_0^2 + C\|\nabla\rho^0\|_1^2 + C\|\nabla\rho^{k+1}\|_1^2 \\ &\le C \max_{0\le t\le T} |\rho_0(t)|^2 |\mathbb{T}^2| + C\delta^2 L_2^2\end{aligned}$$

By Denny [8], the following estimate is obtained

$$\begin{aligned}\|\rho^{k+1}\|_{s+1}^2 &\le \|\rho^{k+1}\|_0^2 + C\|\nabla\rho^{k+1}\|_s^2 \\ &\le C \max_{0\le t\le T} |\rho_0(t)|^2 |\mathbb{T}^2| + CL_2^2 + C\delta^2 L_2^2(1+\lambda c_1) \\ &= L_3^2\end{aligned}$$

where L_3 depends on s, c_1, ρ_0, R, $\|f_t\|_{s-1,T}$, $\|f\|_{s+1,T}$, and $\|\mathbf{w}_0\|_s$. Next by Lemma B.1,

$$\begin{aligned}|\rho^{k+1} - \rho^0|_{L^\infty,T} &\le C\|\nabla(\rho^{k+1} - \rho^0)\|_{1,T} \\ &\le C\|\nabla\rho^{k+1}\|_{s-1,T} \\ &\le C\delta L_2 \\ &< R\end{aligned}$$

and $\rho_0 - R > 0$ for δ sufficiently small.

By definition $|\rho^{k+1} - \rho^0|_{L^\infty,T} < R$ implies $\rho^{k+1} \in \bar{G}_1$ for $x \in \mathbb{T}^n$ and $0 \le t \le T$ which completes the proof.

5.2 Contraction in Low Norm

We will now prove that the sequences are contractive. We start with equation (4.6)

$$\begin{aligned}\mathbf{w}_t^{k+1} + \nabla\phi_t + (\mathbf{w}^k + \nabla\phi)\cdot\nabla(\mathbf{w}^{k+1} + \nabla\phi) + \frac{\lambda p'(\rho^k)}{\rho^k}\nabla\rho^{k+1} \\ = \frac{1}{Re\rho^k}(\Delta\mathbf{w}^{k+1} + (1+\frac{\gamma}{\nu})\nabla\Delta\phi) + c\nabla\Delta\rho^{k+1}\end{aligned}$$

where $\nabla \cdot \mathbf{w}^{k+1} = 0$, $\rho^{k+1}(\mathbf{x}_0, t) = \rho_0(t)$ where $\rho_0(t)$ is a strictly positive function, and $\mathbf{w}^{k+1}(\mathbf{x}, 0) = \mathbf{w}_0(\mathbf{x})$ and where p is a smooth function. We subtract successive iterations to obtain

$$(\mathbf{w}^{k+1} - \mathbf{w}^k)_t + (\mathbf{w}^k + \nabla\phi) \cdot \nabla(\mathbf{w}^{k+1} - \mathbf{w}^k) + \frac{\lambda p'(\rho^k)}{\rho^k} \nabla(\rho^{k+1} - \rho^k)$$
$$= \frac{1}{Re\rho^k} \Delta(\mathbf{w}^{k+1} - \mathbf{w}^k) + c\nabla\Delta(\rho^{k+1} - \rho^k) + \mathbf{h} \qquad (5.3)$$

where

$$\mathbf{h} = (\mathbf{w}^{k-1} - \mathbf{w}^k) \cdot \nabla\mathbf{w}^k + (\frac{\lambda p'(\rho^{k-1})}{\rho^{k-1}} - \frac{\lambda p'(\rho^k)}{\rho^k}) \nabla\rho^k + (\frac{1}{Re\rho^k} - \frac{1}{Re\rho^{k-1}}) \Delta\mathbf{w}^k$$
$$- (\mathbf{w}^k - \mathbf{w}^{k-1}) \cdot \nabla\nabla\phi + (\frac{1}{Re\rho^k} - \frac{1}{Re\rho^{k-1}})(1 + \frac{\gamma}{\nu}) \nabla\Delta\phi$$

We will now prove

$$\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{3,T}^2 + \frac{c_2}{Re} \int_0^T \|D(\mathbf{w}^{k+1} - \mathbf{w}^k)\|_3 d\tau \to 0 \text{ as } k \to \infty$$
$$\|\rho^{k+1} - \rho^k\|_{4,T}^2 \to 0 \text{ as } k \to \infty$$

Proof:

Taking the divergence of equation (5.3) gives

$$c\Delta^2(\rho^{k+1} - \rho^k) - \nabla \cdot [\frac{\lambda p'(\rho^k)}{\rho^k} \nabla(\rho^{k+1} - \rho^k)]$$
$$= \nabla \cdot ((\mathbf{w}^k + \nabla\phi) \cdot \nabla(\mathbf{w}^{k+1} - \mathbf{w}^k)) - \nabla \cdot [\frac{1}{Re\rho^k} \Delta(\mathbf{w}^{k+1} - \mathbf{w}^k)] - \nabla \cdot \mathbf{h}$$

and by Lemma B.1 and Lemma B.2 for r=2

$$c\|\Delta(\rho^{k+1} - \rho^k)\|_2^2 + \lambda c_1 \|\nabla(\rho^{k+1} - \rho^k)\|_2^2 \leq \frac{C}{\lambda c_1} [\|D(\frac{1}{Re\rho^k})\|_3^2 \|\Delta(\mathbf{w}^{k+1} - \mathbf{w}^k)\|_1^2$$
$$+ \|(\mathbf{w}^k + \nabla\phi) \cdot \nabla(\mathbf{w}^{k+1} - \mathbf{w}^k)\|_2^2 + \|\mathbf{h}\|_2^2]$$
$$\leq \frac{C}{\lambda c_1} [\|D(\frac{1}{Re\rho^k})\|_3^2 \|\mathbf{w}^{k+1} - \mathbf{w}^k\|_3^2$$
$$+ \|\mathbf{w}^k + \nabla\phi\|_2^2 \|\mathbf{w}^{k+1} - \mathbf{w}^k\|_3^2 + \|\mathbf{h}\|_2^2] \qquad (5.4)$$

Inequality (5.4) can be rewritten as

$$
\begin{aligned}
c\|\Delta(\rho^{k+1}-\rho^k)\|_2^2 + \lambda c_1\|\nabla(\rho^{k+1}-\rho^k)\|_2^2 \\
&\le \frac{C}{\lambda c_1}[(\frac{\delta^2 L_2^2 C_3}{Re^2} + L_1^2 + \|f\|_1^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_3^2 + \|\mathbf{h}\|_2^2] \\
&\le C_4\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_3^2 + \frac{C}{\lambda c_1}\|\mathbf{h}\|_2^2
\end{aligned}
$$

where C_4 depends on c_1, s, $\|f_t\|_{s-1,T}$, $\|f\|_{s+1,T}$, $\|\mathbf{w}_0\|_s$. Next, we estimate $\|\mathbf{h}\|_2^2$. By Lemma B.1, we have

$$
\begin{aligned}
\|\mathbf{h}\|_2^2 &= \|(\mathbf{w}^{k-1}-\mathbf{w}^k)\cdot\nabla\mathbf{w}^k + (\frac{\lambda p'(\rho^{k-1})}{\rho^{k-1}} - \frac{\lambda p'(\rho^k)}{\rho^k})\nabla\rho^k + (\frac{1}{Re\rho^k} - \frac{1}{Re\rho^{k-1}})\Delta\mathbf{w}^k \\
&\quad - (\mathbf{w}^k-\mathbf{w}^{k-1})\cdot\nabla\nabla\phi + (\frac{1}{Re\rho^k} - \frac{1}{Re\rho^{k-1}})(1+\frac{\gamma}{\nu})\cdot\nabla\nabla\phi\|_2^2 \\
&\le C\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_2^2\|\nabla\mathbf{w}^k\|_2^2 + C\|\frac{\lambda p'(\rho^k)}{\rho^k} - \frac{\lambda p'(\rho^{k-1})}{\rho^{k-1}}\|_2^2\|\nabla\rho^k\|_2^2 \\
&\quad + \frac{C}{Re^2}\|\frac{1}{\rho^k} - \frac{1}{\rho^{k-1}}\|_2^2\|\nabla\mathbf{w}^k\|_3^2 + C\|\nabla\nabla\phi\|_2^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_2^2 \\
&\quad + \frac{C}{Re^2}(1+\frac{\gamma}{\nu})^2\|\nabla\Delta\phi\|_2^2\|\frac{1}{\rho^k} - \frac{1}{\rho^{k-1}}\|_2^2
\end{aligned}
$$

Applying the high norm estimates gives

$$
\begin{aligned}
\|\mathbf{h}\|_2^2 &\le C\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_2^2 L_1^2 + \lambda^2 C_5\|\rho^k-\rho^{k-1}\|_2^2\delta^2 L_2^2 \\
&\quad + \frac{C_5}{Re^2}\|\rho^k-\rho^{k-1}\|_2^2 L_1^2 + C_6\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_2^2 \\
&\quad + \frac{C_7}{Re^2}\|\rho^k-\rho^{k-1}\|_2^2 \\
&\le (CL_1^2+C_6)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_2^2 + \delta^2 C(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)\|\nabla(\rho^k-\rho^{k-1})\|_1^2
\end{aligned}
$$

where C_5 depends on R, ρ_0, c_1, $\|f_t\|_{s-1,T}$, $\|f\|_{s+1,T}$, $\|\mathbf{w}_0\|_s$, C_6 depends on $\|f\|_{2,T}$, C_7 depends on γ, ν, R, ρ_0, c_1, $\|f_t\|_{s-1,T}$, $\|f\|_{s+1,T}$, $\|\mathbf{w}_0\|_s$, and $\frac{1}{Re^2} \le \delta^2$, and we used the inequalities in Lemma B.1. Therefore,

$$
\begin{aligned}
c\|\Delta(\rho^{k+1}-\rho^k)\|_2^2 + \lambda c_1\|\nabla(\rho^{k+1}-\rho^k)\|_2^2 \\
&\le C_4\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_3^2 + \frac{C}{\lambda c_1}(L_1^2+C_6)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_3^2 \\
&\quad + \frac{C}{\lambda c_1}\delta^2(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)\|\nabla(\rho^k-\rho^{k-1})\|_2^2 \qquad (5.5)
\end{aligned}
$$

Using Lemma B.3 with r=3, we get

$$\begin{aligned}
\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_3^2 &+ \frac{c_2}{Re}\int_0^T \|D(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_3^2 d\tau \\
&\le C\int_0^T \|\mathbf{h}\|_3^2 d\tau \\
&\le C\int_0^T ((L_1^2+C_6)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_3^2 \\
&\qquad + \delta^2(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)\|\nabla(\rho^k-\rho^{k-1})\|_2^2) d\tau \qquad (5.6)
\end{aligned}$$

Now, we add (5.5) and (5.6) to get

$$\begin{aligned}
&\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{3,T}^2 + \frac{c_2}{Re}\int_0^T \|D(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_3^2 d\tau + \beta(c\|\Delta(\rho^{k+1}-\rho^k)\|_{2,T}^2 + \lambda c_1\|\nabla(\rho^{k+1}-\rho^k)\|_{2,T}^2) \\
&\quad \le CT(L_1^2+C_6)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_{3,T}^2 + T\delta^2 C(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)\|\nabla(\rho^k-\rho^{k-1})\|_{2,T}^2 \\
&\qquad + \beta C_4\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{3,T}^2 + \beta\frac{C}{\lambda c_1}(L_1^2+C_6)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_{3,T}^2 \\
&\qquad + \beta\frac{C}{\lambda c_1}\delta^2(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)\|\nabla(\rho^k-\rho^{k-1})\|_{2,T}^2
\end{aligned}$$

If $\beta C_4 \le \frac{1}{2}$ the above inequality becomes

$$\begin{aligned}
&\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_3^2 + \frac{c_2}{Re}\int_0^T \|D(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_3^2 d\tau + 2\beta(c\|\Delta(\rho^{k+1}-\rho^k)\|_{2,T}^2 + \lambda c_1\|\nabla(\rho^{k+1}-\rho^k)\|_{2,T}^2) \\
&\quad \le 2CT(L_1^2+C_6)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_{3,T}^2 + 2T\delta^2 C(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)\|\nabla(\rho^k-\rho^{k-1})\|_{2,T}^2 \\
&\qquad + 2\beta\frac{C}{\lambda c_1}(L_1^2+C_6)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_{3,T}^2 + 2\beta\frac{C}{\lambda c_1}\delta^2(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)\|\nabla(\rho^k-\rho^{k-1})\|_{2,T}^2 \\
&\quad \le 2(L_1^2+C_6)(TC+\beta\frac{C}{\lambda c_1})[\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_{3,T}^2 + \frac{c_2}{Re}\int_0^T \|D(\mathbf{w}^k-\mathbf{w}^{k-1})\|_3 d\tau] \\
&\qquad + 2(\lambda^2 C_5 L_2^2 + C_5 L_1^2 + C_7)(CT\delta^2 + \frac{C}{\lambda c_1}\beta\delta^2)[c\|\Delta(\rho^k-\rho^{k-1})\|_{2,T}^2 \\
&\qquad + \lambda c_1\|\nabla(\rho^k-\rho^{k-1})\|_{2,T}^2]
\end{aligned}$$

If T, β, and δ^2 are sufficiently small then $2(L_1^2+C_6)(TC+\beta\frac{C}{\lambda c_1}) < \xi$ and $2(C_5\lambda^2 L_2^2 + C_5 L_1^2 + C_7)(T\delta^2 C + \frac{C}{\lambda c_1}\beta\delta^2) < \xi(2\beta)$ where $\xi < 1$. This gives us the contraction

$$\begin{aligned}
&\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{3,T}^2 + \frac{c_2}{Re}\int_0^T \|D(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_3^2 d\tau + 2\beta[c\|\Delta(\rho^{k+1}-\rho^k)\|_{2,T}^2 + \lambda c_1\|\nabla(\rho^{k+1}-\rho^k)\|_{2,T}^2] \\
&\quad \le \xi[\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_{3,T}^2 + \frac{c_2}{Re}\int_0^T \|D(\mathbf{w}^k-\mathbf{w}^{k-1})\|_3^2 d\tau \\
&\qquad + 2\beta[c\|\Delta(\rho^k-\rho^{k-1})\|_{2,T}^2 + \lambda c_1\|\nabla(\rho^k-\rho^{k-1})\|_{2,T}^2]]
\end{aligned}$$

Therefore,

$$\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{3,T}^2+\frac{c_2}{Re}\int_0^T\|D(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_3^2d\tau\to 0 \text{ as } k\to\infty \tag{5.7}$$

and

$$c\|\Delta(\rho^{k+1}-\rho^k)\|_{2,T}^2+\lambda c_1\|\nabla(\rho^{k+1}-\rho^k)\|_{2,T}^2\to 0 \text{ as } k\to\infty \tag{5.8}$$

Therefore by (5.8) and by the inequality

$$\|\nabla(\rho^{k+1}-\rho^k)\|_{3,T}^2\leq C(c\|\Delta(\rho^{k+1}-\rho^k)\|_{2,T}^2+\lambda c_1\|\nabla(\rho^{k+1}-\rho^k)\|_{2,T}^2)$$

it follows that

$$\|\nabla(\rho^{k+1}-\rho^k)\|_{3,T}^2\to 0 \text{ as } k\to\infty$$

By Lemma B.1

$$\|\rho^{k+1}-\rho^k\|_{4,T}^2\to 0 \text{ as } k\to\infty$$

which completes the proof.

5.3 Convergence to Solution of Nonlinear Equations

We will now prove the convergence of the sequence of solutions $\{\mathbf{w}^k\}$, $\{\rho^k\}$ to a unique solution of the nonlinear system. From (5.7) and (5.8) we conclude that $\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{3,T}^2\to 0$ and $\|\rho^{k+1}-\rho^k\|_{4,T}^2\to 0$ as $k\to\infty$. Therefore, there exist $\mathbf{w}\in C([0,T],H^3(\mathbb{T}^2))$ and $\rho\in C([0,T],H^4(\mathbb{T}^2))$ such that $\|\mathbf{w}^k-\mathbf{w}\|_{3,T}^2\to 0$ and $\|\rho^k-\rho\|_{4,T}^2\to 0$ as $k\to\infty$. Using the standard interpolation inequality $\|f\|_{s'}\leq C\|f\|_3^{\alpha}\|f\|_s^{1-\alpha}$, with $\alpha=\frac{s-s'}{s-3}$, and $\|g\|_{s'+1}\leq C\|g\|_4^{\beta}\|g\|_{s+1}^{1-\beta}$, with $\beta=\frac{s-s'}{s-3}$, with $s'<s$, we conclude that $\|\mathbf{w}^k-\mathbf{w}\|_{s',T}\to 0$ and $\|\rho^k-\rho\|_{s'+1,T}\to 0$ as $k\to\infty$ for any $s'<s$. For $s'>\frac{N}{2}+4=5$ Sobolev's lemma implies that $\mathbf{w}^k\to\mathbf{w}\in C([0,T],C^4(\mathbb{T}^2))$ and $\rho^k\to\rho\in C([0,T],C^5(\mathbb{T}^2))$. Since $\rho^k(\mathbf{x}_0,t)=\rho_0(t)$ for all k, $\rho(\mathbf{x}_0,t)=\rho_0(t)$. From the linear system of equations (4.6) it follows that

$\|\mathbf{w}_t^k - \mathbf{w}_t\|_{s'-2,T} \to 0$ as $k \to \infty$, so that $\mathbf{w}_t^k \to \mathbf{w}_t \in C([0,T], C^2(\mathbb{T}^2))$. Since ρ^{k+1}, $\mathbf{w}^{k+1}$ is a solution of (4.6)-(4.8) for $k \geq 0$, it follows that ρ, $\mathbf{w}$ is a classical solution of (4.3)-(4.5).

We will now prove uniqueness of a solution by examining two solutions to equations (4.3), (4.4). We will denote the solutions as $\mathbf{w}_1$, $\mathbf{w}_2$, ρ_1, and ρ_2. The solutions have the regularity outlined above and satisfy

$$\mathbf{w}_1(\mathbf{x},0) = \mathbf{w}_2(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x})$$
$$\rho_1(\mathbf{x}_0,t) = \rho_2(\mathbf{x}_0,t) = \rho_0(t)$$

To prove uniqueness we must show that $\rho_1 = \rho_2$ and $\mathbf{w}_1 = \mathbf{w}_2$. We start by determining the differences of equation (4.3)

$$(\mathbf{w}_1 - \mathbf{w}_2)_t + (\mathbf{w}_1 + \nabla\phi)\cdot\nabla(\mathbf{w}_1 - \mathbf{w}_2) + \frac{\lambda p'(\rho_1)\nabla(\rho_1 - \rho_2)}{\rho_1}$$
$$= \frac{1}{\rho_1 Re}\Delta(\mathbf{w}_1 - \mathbf{w}_2) + c\nabla\Delta(\rho_1 - \rho_2) + \mathbf{F}$$

where

$$\mathbf{F} = (\mathbf{w}_2 - \mathbf{w}_1)\cdot\nabla\mathbf{w}_2 + (\mathbf{w}_2 - \mathbf{w}_1)\cdot\nabla\nabla\phi + (\frac{\lambda p'(\rho_2)}{\rho_2} - \frac{\lambda p'(\rho_1)}{\rho_1})\nabla\rho_2$$
$$+ (\frac{1}{\rho_1 Re} - \frac{1}{\rho_2 Re})\Delta\mathbf{w}_2 + (\frac{1}{(\rho_1 Re} - \frac{1}{\rho_2 Re})(1 + \frac{\gamma}{\nu})\nabla\Delta\phi$$

By using the contraction estimate, it follows that

$$\|\mathbf{w}_1 - \mathbf{w}_2\|_{3,T}^2 + \frac{c_2}{Re}\int_0^T \|D(\mathbf{w}_1 - \mathbf{w}_2)\|_3^2 d\tau + 2\beta[\lambda c_1\|\nabla(\rho_1 - \rho_2)\|_{2,T}^2 + c\|\Delta(\rho_1 - \rho_2)\|_{2,T}^2] = 0$$

and $\|\rho_1 - \rho_2\|_{4,T}^2 = 0$ by Lemma B.1. This completes the proof of the main theorem.

CHAPTER VI: CONCLUSION

In conclusion, we have proved the existence of a unique solution to the system of equations (3.15) and (3.16) under periodic boundary conditions for the time interval $0 \leq t \leq T$. The data for the system consists of an initial velocity, density data at a point in the spatial domain, and the divergence of velocity. This system of equations models a compressible barotropic fluid. The equations are modified versions of the compressible Navier-Stokes equations with the inclusion of a capillary stress term. Specifically, the main new result of this thesis is the use of density data at a given point in the spatial domain instead of using initial density. The methodology utilized in this thesis is the method of successive approximations. We defined an iteration scheme based on solving a linearized version of the equations, then proved convergence of the sequence of approximate solutions to a unique solution of the nonlinear system. This methodology can be applied to future research problems, such as fluid flow problems where the data such as the concentration of a solute (such as a pollutant) in a fluid or the temperature of the fluid is known at a given point. Other possible future research could use boundary conditions which are not periodic.

[1] D. M. Anderson, G. B. McFadden and A. A. Wheeler, *Diffuse-Interface Methods In Fluid Mechanics*. Annual Review of Fluid Mechanics, 30, No. 1, (1998), 139-165.

[2] D. Bresch, B. Desjardins and D. Gerard-Varet, *On Compressible Navier-Stokes Equations with Density Dependent Viscosities in Bounded Domains*, J. Math. Pure Appl., No. 87 (2007), 227-235.

[3] H. J. Choe and H. Kim, *Strong Solutions of the Navier-Stokes Equations for Isentropic Compressible Fluids*, J. Differential Equation, No. 190 (2003), 504-523.

[4] A. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer, 2000.

[5] D. Denny, *Existence of a Unique Solution to a Quasilinear Elliptic Equation*, Journal of Mathematical Analysis and Applications, 380, (2011), 653-668.

[6] D. Denny, *A Unique Solution to a Nonlinear Elliptic Equation*, Journal of Mathematical Analysis and Applications, 365, (2010), 467-482.

[7] D. Denny, *Existence and Uniqueness of Classical Solutions to Second-order Quasilinear Elliptic Equations*, Electronic Journal of Differential Equations, 2010, No. 82, (2010), 1-21.

[8] D. Denny, *Existence and Uniqueness of Global Solutions to a model for the flow of an Incompressible, Barotropic Fluid with Capillary Effects*, Electronic Journal of Differential Equations, 2007, No. 39, (2007), 1-23.

[9] D. Denny and R. Pego, *Models of Low-Speed Flow for Near-Critical Fluids with Gravitational and Capillary Effects*, Quarterly of Applied Mathematics, 58, (2000), 103-125.

[10] B. Desjardins, *Regularity of Weak Solutions of the Compressible Isentropic Navier-Stokes Equations*, Comm. in Partial Differential Equations 22, No. 5 and 6, (1997), 977-1008.

[11] P. Embid, *On the Reactive and Non-diffusive Equations for Zero Mach number Combustion*, Comm. in Partial Differential Equations 14, No. 8 and 9, (1989), 1249-1281.

[12] P. Embid, *Well-posedness of the Nonlinear Equations for Zero Mach number Combustion*, Comm. in Partial Differential Equations 12, No. 11, (1987), 1227-1283.

[13] L. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics 19, American Mathematical Society, Providence, Rhode Island, 1998.

[14] D. Hoff, *Strong Convergence to Global Solutions for Multidimensional Flows of Compressible, Viscous Fluids with Polytropic Equations of State and Discontinuous Initial Data*, Arch. Rational Mech. Anal., No. 132, (1995), 1-14

[15] D. Hoff, *Global Solutions of the Navier-Stokes Equations for Multidimensional Compressible Flow with Discontinuous Initial Data*, J. Differential Equations, No. 120, (1995), 215-254.

[16] A. Mellet and A. Vasseur, *On the Barotropic Compressible Navier-Stokes Equations*, Comm. in Partial Differential Equations, No. 32, (2007), 43.

[17] S. Song, *Existence of a Unique Solution to a System of Equations Modeling Compressible Fluid Flow*, Master's Thesis, Texas A & M University - Corpus Christi, Spring 2015.

APPENDIX A: PROOF OF LINEAR EXISTENCE

We now prove the existence of a solution to the linear system of equations (4.6)-(4.10) for each fixed k. The proof of the following lemma appears in Denny [8].

Lemma A.1

Given $a \in C([0,T],H^0(\Omega)) \cap L^\infty([0,T],H^{s-1}(\Omega))$ *and* $\mathbf{f} \in C([0,T],H^0(\Omega)) \cap L^\infty([0,T],H^{s-2}(\Omega))$, *where* $s > \frac{N}{2}+4$, $a(\mathbf{x},t) \geq c_1$, *with* $c_1 > 0$ *for* $\mathbf{x} \in \Omega$, $\Omega = \mathbb{T}^N$, $0 \leq t \leq T$, *there is a classical solution* $\theta \in C([0,T],C^5(\Omega)) \cap L^\infty([0,T],H^{s+1}(\Omega))$ *of*

$$c\Delta^2\theta - \lambda\nabla\cdot(a\nabla\theta) = \nabla\cdot\mathbf{f}$$

Here c is a positive constant.

The next lemma proves the existence of a solution to the linear system of equations.

Lemma A.2

Given $f \in C([0,T],C^5(\Omega)) \cap L^\infty([0,T],H^{s+1}(\Omega))$, $f_t \in L^\infty([0,T],H^{s-1}(\Omega))$, $\int_\Omega f dx = 0$, $r \in C([0,T],C^5(\Omega)) \cap L^\infty([0,T],H^{s+1}(\Omega))$, *and* $q \in C([0,T],C^4(\Omega)) \cap L^\infty([0,T],H^s(\Omega))$, *and given* $s > \frac{N}{2}+4$, $a(\mathbf{x},t) \geq c_1$, *with* $c_1 > 0$ *for* $\mathbf{x} \in \Omega$, $\Omega = \mathbb{T}^N$, $0 \leq t \leq T$, *there is a solution* $u \in C([0,T],C^4(\Omega)) \cap L^\infty([0,T],H^s(\Omega)) \cap L^2([0,T],H^{s+1}(\Omega))$, $h \in C([0,T],C^5(\Omega)) \cap L^\infty([0,T],H^{s+1}(\Omega))$ *of*

$$\mathbf{u}_t + \nabla\phi_t + (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{u}+\nabla\phi) + \frac{\lambda p'(r)}{r}\nabla h = \frac{1}{rRe}(\Delta\mathbf{u} + (1+\frac{\gamma}{\nu})\nabla\Delta\phi) + c\nabla\Delta h \tag{A.2}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{A.3}$$

$$\Delta\phi = f \tag{A.4}$$

$$h(\mathbf{x}_0, t) = \rho_0(t) \tag{A.5}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{w}_0(\mathbf{x}) \tag{A.6}$$

Here, c is a positive constant.

Proof: We will now apply the operators P and Q which perform the orthogonal projection of $L^2(\mathbb{T}^2)$ onto the solenoidal vector field and the gradient vector field [4]. Equation (A.2) becomes after projection by P

$$\begin{aligned}\mathbf{u}_t + (\mathbf{q} + \nabla\phi) \cdot \nabla(\mathbf{u} + \nabla\phi) - Q[(\mathbf{q} + \nabla\phi) \cdot \nabla(P\mathbf{u} + \nabla\phi)] + P[\lambda \frac{p^{'}(r)\nabla h}{r}] \\ = \frac{1}{rRe}\Delta\mathbf{u} - Q[\frac{1}{rRe}\Delta P\mathbf{u}] + P[\frac{1}{rRe}(1 + \frac{\gamma}{\nu})\nabla\Delta\phi)]\end{aligned} \tag{A.7}$$

and under projection by Q

$$\begin{aligned}Q[\mathbf{u}_t + \nabla\phi_t + Q((\mathbf{q} + \nabla\phi) \cdot \nabla(P\mathbf{u} + \nabla\phi)) + \lambda \frac{p^{'}(r)\nabla h}{r} \\ - Q(\frac{1}{rRe}\Delta P\mathbf{u}) - \frac{1}{rRe}(1 + \frac{\gamma}{\nu})\nabla\Delta\phi - c\nabla\Delta h] = 0\end{aligned}$$

which is equivalent to

$$\begin{aligned}\nabla \cdot [\mathbf{u}_t + \nabla\phi_t + Q((\mathbf{q} + \nabla\phi) \cdot \nabla(P\mathbf{u} + \nabla\phi)) + \lambda \frac{p^{'}(r)\nabla h}{r} \\ - Q(\frac{1}{rRe}\Delta P\mathbf{u}) - \frac{1}{rRe}(1 + \frac{\gamma}{\nu})\nabla\Delta\phi - c\nabla\Delta h] = 0\end{aligned}$$

Since $\nabla \cdot \mathbf{u} = 0$, the previous equation can be rewritten as

$$\begin{aligned}\Delta\phi_t + \nabla \cdot [Q((\mathbf{q} + \nabla\phi) \cdot \nabla(P\mathbf{u} + \nabla\phi))] + \nabla \cdot [\lambda \frac{p^{'}(r)\nabla h}{r}] \\ - \nabla \cdot [Q(\frac{1}{rRe}\Delta P\mathbf{u})] - \nabla \cdot [\frac{1}{rRe}(1 + \frac{\gamma}{\nu})\nabla\Delta\phi] - c\Delta^2 h = 0\end{aligned}$$

Now rearranging the terms yields

$$\begin{aligned}c\Delta^2 h - \nabla \cdot [\lambda \frac{p^{'}(r)\nabla h}{r}] = f_t + \nabla \cdot [Q((\mathbf{q} + \nabla\phi) \cdot \nabla(P\mathbf{u} + \nabla\phi))] \\ - \nabla \cdot [Q(\frac{1}{rRe}\Delta P\mathbf{u})] - \nabla \cdot [\frac{1}{rRe}(1 + \frac{\gamma}{\nu})\nabla f]\end{aligned} \tag{A.8}$$

where $\Delta\phi = f$. We will next prove the existence of a solution $\mathbf{u}$ and h to equations (A.7) and (A.8). In order to prove the existence of a solution, we will use a sequence of approximate solutions for $\mathbf{u}$ and h. Let $\mathbf{u}^0 = \mathbf{w}_0$ and $h^0 = \rho_0$ be the initial iterates of $\mathbf{u}^k$ and h^k respectively. We will use the following iteration scheme

$$\begin{aligned}
\mathbf{u}_t^{k+1} + (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{u}^{k+1}+\nabla\phi) - \frac{1}{rRe}\Delta\mathbf{u}^{k+1} \\
= Q[(\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi)] - P[\lambda\frac{p'(r)\nabla h^{k+1}}{r}] \\
- Q[\frac{1}{rRe}\Delta P\mathbf{u}^k] + P[\frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f)] \qquad \text{(A.9)}
\end{aligned}$$

$$\begin{aligned}
c\Delta^2 h^{k+1} - \nabla\cdot[\lambda\frac{p'(r)\nabla h^{k+1}}{r}] = f_t + \nabla\cdot[Q((\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi))] \\
- \nabla\cdot[Q(\frac{1}{rRe}\Delta P\mathbf{u}^k)] - \nabla\cdot[\frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f] \qquad \text{(A.10)}
\end{aligned}$$

The proof of the existence of a solution h^{k+1} to equation (A.10) appears in Lemma A.1 where we take

$$\begin{aligned}
\mathbf{f} &= \nabla\phi_t + Q((\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi)) - Q(\frac{1}{rRe}\Delta P\mathbf{u}^k) - \frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f \\
\theta &= h^{k+1} \\
a &= \frac{p'(r)}{r}
\end{aligned}$$

where $a \in C([0,T],H^0)\cap L^\infty([0,T],H^{s+1}(\Omega))$.

And $\mathbf{f} \in C([0,T],H^0) \cap L^\infty([0,T],H^{s-1}(\Omega))$ because

$$\begin{aligned}
\|\mathbf{f}\|_{s-1} &= \|\nabla\phi_t + Q((\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi)) - Q(\frac{1}{rRe}\Delta P\mathbf{u}^k) - \frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f\|_{s-1} \\
&\le C\|\nabla\phi_t\|_{s-1} + C\|Q((\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi))\|_{s-1} \\
&\quad + C\|Q(\frac{1}{rRe}\Delta P\mathbf{u}^k)\|_{s-1} + C\|\frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f\|_{s-1} \\
&\le C\|\nabla\phi_t\|_{s-1} + C\|\mathbf{q}+\nabla\phi\|_{s-1}\|\nabla(P\mathbf{u}^k+\nabla\phi))\|_{s-1} \\
&\quad + C\|D(\frac{1}{rRe})\|_{s-2}\|P\mathbf{u}^k\|_s + C\|\frac{1}{rRe}(1+\frac{\gamma}{\nu})\|_{s-1}\|\nabla f\|_{s-1} \\
&\le C\|\nabla\phi_t\|_{s-1} + C(\|\mathbf{q}\|_{s-1}+\|\nabla\phi\|_{s-1})(\|\nabla(P\mathbf{u}^k)\|_{s-1}+\|\nabla\nabla\phi\|_{s-1}) \\
&\quad + C\|D(\frac{1}{rRe})\|_{s-2}\|\mathbf{u}^k\|_s + C\|\frac{1}{rRe}(1+\frac{\gamma}{\nu})\|_{s-1}\|\nabla f\|_{s-1} \\
&\le C\|\nabla\phi_t\|_{s-1} + C(\|\mathbf{q}\|_{s-1}+\|\nabla\phi\|_{s-1})(\|\mathbf{u}^k\|_s+\|\nabla\phi\|_s) \\
&\quad + C\|\frac{1}{rRe}\|_{s-1}\|\mathbf{u}^k\|_s + C\|\frac{1}{rRe}(1+\frac{\gamma}{\nu})\|_{s-1}\|f\|_s
\end{aligned} \tag{A.11}$$

where $\|\nabla\phi_t\|_{s-1} \le C\|f_t\|_{s-2}$ and $\|\nabla\phi\|_{s-1} \le C\|f\|_{s-2}$ by Lemma B.1. It follows that

$$\begin{aligned}
\|\nabla h^{k+1}\|_{s-1}^2 \le \frac{C}{\lambda^2 c_1^2}[\|f_t\|_{s-2}^2 + (\|\mathbf{q}\|_{s-1}^2+\|f\|_{s-2}^2)(\|\mathbf{u}^k\|_s^2+\|f\|_{s-1}^2) \\
+ \|\frac{1}{rRe}\|_{s-1}^2\|\mathbf{u}^k\|_s^2 + \|\frac{1}{rRe}(1+\frac{\gamma}{\nu})\|_{s-1}^2\|f\|_s^2]
\end{aligned} \tag{A.12}$$

by Lemma B.2. We will begin the proof that equation (A.9) has a solution for each fixed k by the Galerkin method. We begin this method by letting $\mathbf{g} = \mathbf{u}^{k+1}$. We choose an orthonormal basis $\Phi_i(x,y)$, for $i = 1,2,3,...$ for $L^2(\mathbb{T}^2)$ and define the projection operator P^m so that it approximates any function with a linear combination of the first m components in the chosen basis. Using this basis we approximate $\mathbf{g}$ with

$$\begin{aligned}
P^m\mathbf{g} &= \mathbf{g}^m \\
&= \Sigma_{i=1}^m \vec{\alpha}_i(t)\Phi_i(x,y)
\end{aligned} \tag{A.13}$$

We use a linear system of differential equations to find each $\mathbf{g}^m$. We start with equation (A.9)

$$\mathbf{u}_t^{k+1} + (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{u}^{k+1}+\nabla\phi) - \frac{1}{rRe}\Delta\mathbf{u}^{k+1} = \mathbf{F}^k \tag{A.14}$$

where $\mathbf{F}^k = Q[(\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi)] - P[\lambda\frac{p'(r)\nabla h^{k+1}}{r}] - Q[\frac{1}{rRe}\Delta P\mathbf{u}^k] + P[\frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f)]$. Since $\mathbf{g} = \mathbf{u}^{k+1}$ equation (A.14) becomes

$$\mathbf{g}_t + (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}+\nabla\phi) - \frac{1}{rRe}\Delta\mathbf{g} = \mathbf{G}$$

where $\mathbf{G} = \mathbf{F}^k$. Applying the operator P^m yields

$$\mathbf{g}_t^m + P^m((\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}^m+\nabla\phi)) - P^m(\frac{1}{rRe}\Delta\mathbf{g}^m) = P^m\mathbf{G} \tag{A.15}$$

Plugging in equation (A.13) into (A.15) gives

$$\Sigma_{i=1}^m \dot{\vec{\alpha}}_i(t)\Phi_i + P^m((\mathbf{q}+\nabla\phi)\cdot\nabla(\Sigma_{i=1}^m \vec{\alpha}_i(t)\Phi_i+\nabla\phi)) - P^m(\frac{1}{rRe}\Sigma_{i=1}^m \vec{\alpha}_i(t)\Delta\Phi_i) \\ = P^m\mathbf{G}$$

We take the L^2 inner product with Φ_j, where $1 \leq j \leq m$, to get

$$(\Sigma_{i=1}^m \dot{\vec{\alpha}}_i(t)\Phi_i + P^m((\mathbf{q}+\nabla\phi)\cdot\nabla(\Sigma_{i=1}^m \vec{\alpha}_i(t)\Phi_i+\nabla\phi)) - P^m(\frac{1}{rRe}\Sigma_{i=1}^m \vec{\alpha}_i(t)\Delta\Phi_i),\Phi_j) \\ = (P^m\mathbf{G},\Phi_j) \tag{A.16}$$

Simplifying equation (A.16) gives

$$\dot{\vec{\alpha}}_j = \frac{1}{Re}\Sigma_{i=1}^m \vec{\alpha}_i(P^m(\frac{1}{r}\Delta\Phi_i),\Phi_j) - (P^m((\mathbf{q}+\nabla\phi)\cdot\nabla(\Sigma_{i=1}^m \vec{\alpha}_i(t)\Phi_i+\nabla\phi)),\Phi_j) + (P^m\mathbf{G},\Phi_j)$$

Each $\vec{\alpha}_j$ can be determined by solving the above system of ordinary differential equations. We will now prove the convergence of $\mathbf{g}^m = \Sigma_{i=1}^m \vec{\alpha}_i(t)\Phi_i(x,y)$ as $m \to \infty$. The first fact utilized is that because $\alpha_i(t)_{i=1}^m \in C^1([0,T])$, $\mathbf{g}^m \in C^1([0,T],H^r)$ for any $r \geq 0$ (see, e.g., Embid [11]). We estimate $\|\mathbf{G}\|_s$ as follows

$$
\begin{aligned}
\|\mathbf{G}\|_s &= \|Q[(\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi)] - P[\lambda\frac{p'(r)\nabla h^{k+1}}{r}] \\
&\quad - Q[\frac{1}{rRe}\Delta P\mathbf{u}^k] + P[\frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f)]\|_s \\
&\le C\|Q[(\mathbf{q}+\nabla\phi)\cdot\nabla(P\mathbf{u}^k+\nabla\phi)]\|_s + C\|P[\lambda\frac{p'(r)\nabla h^{k+1}}{r}]\|_s \\
&\quad + C\|Q[\frac{1}{rRe}\Delta P\mathbf{u}^k\|_s + C\|P[\frac{1}{rRe}(1+\frac{\gamma}{\nu})\nabla f)]\|_s \\
&\le C\|\mathbf{q}+\nabla\phi\|_s(\|\mathbf{u}^k\|_s+\|\nabla\phi\|_{s+1}) + C\|D(\frac{\lambda p'(r)}{r})\|_s\|\nabla h^{k+1}\|_{s-1} \\
&\quad + C\|D(\frac{1}{rRe})\|_{s-1}\|\mathbf{u}^k\|_{s+1} + \|D(\frac{1}{rRe}(1+\frac{\gamma}{\nu}))\|_s\|\nabla f\|_{s-1} \\
&\le C(\|\mathbf{q}\|_s+\|\nabla\phi\|_s)(\|\mathbf{u}^k\|_s+\|\nabla\phi\|_{s+1}) + C\|D(\frac{\lambda p'(r)}{r})\|_s\|\nabla h^{k+1}\|_{s-1} \\
&\quad + C\|D(\frac{1}{rRe})\|_{s-1}(\|\mathbf{u}^k\|_0+\|D\mathbf{u}^k\|_s) + \|D(\frac{1}{rRe}(1+\frac{\gamma}{\nu}))\|_s\|f\|_s \\
&\le C(\|\mathbf{q}\|_s+\|\nabla\phi\|_s)(\|\mathbf{u}^k\|_s+\|\nabla\phi\|_{s+1}) + C\|D(\frac{\lambda p'(r)}{r})\|_s\|\nabla h^{k+1}\|_{s-1} \\
&\quad + C\|D(\frac{1}{rRe})\|_{s-1}(\|\mathbf{u}^k\|_s+\|D\mathbf{u}^k\|_s) + \|D(\frac{1}{rRe}(1+\frac{\gamma}{\nu}))\|_s\|f\|_s
\end{aligned}
$$

where $\nabla h^{k+1} \in L^\infty([0,T], H^{s-1}(\Omega)$ by (A.12). Next we will derive a Sobolev space estimate for $\mathbf{g}^m$ and use this to prove convergence to $\mathbf{g}$. By applying Lemma B.3 to equation (A.15), we derive the estimate

$$
\|\mathbf{g}^m\|_s^2 + \frac{c_2}{Re}\int_0^t \|D\mathbf{g}^m\|_s^2 d\tau \le C[\|\mathbf{w}_0\|_s^2 + \int_0^t \|\mathbf{G}\|_s^2 d\tau] \tag{A.17}
$$

where C is a constant and $\mathbf{G} \in L^\infty([0,T], H^s(\Omega)) \cap L^2([0,T], H^{s+1}(\Omega))$. Since the right-hand side of inequality (A.17) does not depend on m, and we can take $\|\mathbf{w}_0\|_s^2$ to be bounded, we have from inequality (A.17) that $\|\mathbf{g}^m\|_{s,T}^2 + \frac{c_2}{Re}\int_0^T \|D\mathbf{g}^m\|_s^2 d\tau$ is bounded. This gives a bound in $L^\infty([0,T], H^s(\Omega)) \cap L^2([0,T], H^{s+1}(\Omega))$. We will now show equicontinuity. From equation (A.15), we may derive

$$
\begin{aligned}
\|\mathbf{g}_t^m\|_0^2 &= \|P^m\mathbf{G} - P^m((\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}^m+\nabla\phi)) + P^m(\frac{1}{rRe}\Delta\mathbf{g}^m)\|_0^2 \\
&\le C[\|P^m\mathbf{G}\|_0^2 + \|P^m((\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}^m+\nabla\phi))\|_0^2 + \|P^m(\frac{1}{rRe}\Delta\mathbf{g}^m)\|_0^2] \\
&\le C[\|\mathbf{G}\|_0^2 + |\mathbf{q}+\nabla\phi|_{L^\infty}^2\|\nabla(\mathbf{g}^m+\nabla\phi)\|_0^2 + |\frac{1}{rRe}|_{L^\infty}^2\|\Delta\mathbf{g}^m\|_0^2]
\end{aligned}
$$

This establishes equicontinuity (see, e.g., Embid [11]). Now we use Arzelá -Ascoli theorem and the weak$-*$ compactness of bounded sets in $L^\infty([0,T],H^s(\Omega))$, to conclude that there is a subsequence $\{\mathbf{g}^{m'}\}$ of $\{\mathbf{g}^m\}$ such that $\mathbf{g}^{m'} \to \mathbf{g}$, with $\mathbf{g} \in C([0,T],H^0(\Omega)) \cap L^\infty([0,T],H^s(\Omega)) \cap L^2([0,T],H^{s+1}(\Omega))$. Moreover, since every $\mathbf{g}^{m'}$ satisfies equation (A.17), we can deduce

$$\|\mathbf{g}\|_s^2 + \frac{c_2}{Re}\int_0^t \|D\mathbf{g}\|_s^2 d\tau \leq C[\|\mathbf{w}_0\|_s^2 + \int_0^t \|\mathbf{G}\|_s^2 d\tau]$$

using the method of Embid [11]. Each $\mathbf{g}^{m'}$ satisfies

$$\mathbf{g}^{m'} = P^{m'}\mathbf{w}_0 + \int_0^t P^{m'}(\mathbf{G} - (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}^{m'}+\nabla\phi) + \frac{1}{rRe}\Delta\mathbf{g}^{m'})d\tau \tag{A.18}$$

Since $\mathbf{g}^{m'} \to \mathbf{g}$ and $P^{m'}\mathbf{G} - P^{m'}((\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}^{m'}+\nabla\phi)) + P^{m'}(\frac{1}{rRe}\Delta\mathbf{g}^{m'}))$ is uniformly bounded, $P^{m'}(\mathbf{G} - (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}^{m'}+\nabla\phi) + \frac{1}{rRe}\Delta\mathbf{g}^{m'}) \to \mathbf{G} - (\mathbf{q}+\nabla\phi)\cdot\nabla(\vec{\mathbf{g}}+\nabla\phi) + \frac{1}{rRe}\Delta\vec{\mathbf{g}}$ as $m' \to \infty$ pointwise. By the Lebesgue's dominated convergence theorem, we can therefore conclude from equation (A.18) that

$$\mathbf{g} = \mathbf{w}_0 + \int_0^t (\mathbf{G} - (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{g}+\nabla\phi) + \frac{1}{rRe}\Delta\mathbf{g})d\tau$$

This equation means that $\mathbf{g}$ satisfies equation (A.14). Therefore a solution $\mathbf{u}^{k+1} \in C([0,T],C^4(\Omega)) \cap L^\infty([0,T],H^s(\Omega)) \cap L^2([0,T],H^{s+1}(\Omega))$ to equation (A.14) exists. We will now prove the convergence of the sequences $\{\mathbf{u}^{k+1}\}$, $\{h^{k+1}\}$ to a solution $\mathbf{u}$, h of the linear system of equations. We begin by subtracting the subsequent iterations of equations (A.9) and (A.10)

$$\begin{aligned}(\mathbf{u}^{k+1}-\mathbf{u}^k)_t &+ (\mathbf{q}+\nabla\phi)\cdot\nabla(\mathbf{u}^{k+1}-\mathbf{u}^k) - \frac{1}{rRe}\Delta(\mathbf{u}^{k+1}-\mathbf{u}^k)\\ &= Q[(\mathbf{q}+\nabla\phi)\cdot\nabla(P(\mathbf{u}^k-\mathbf{u}^{k-1}))] - P[\lambda\frac{p'(r)\nabla(h^{k+1}-h^k)}{r}]\\ &\quad - Q[\frac{1}{rRe}\Delta P(\mathbf{u}^k-\mathbf{u}^{k-1})]\end{aligned} \tag{A.19}$$

$$\begin{aligned}c\Delta^2(h^{k+1}-h^k) &- \nabla\cdot[\lambda\frac{p'(r)\nabla(h^{k+1}-h^k)}{r}]\\ &= \nabla\cdot[Q((\mathbf{q}+\nabla\phi)\cdot\nabla(P(\mathbf{u}^k-\mathbf{u}^{k-1}))] - \nabla\cdot[\frac{1}{rRe}\Delta P(\mathbf{u}^k-\mathbf{u}^{k-1})]\end{aligned} \tag{A.20}$$

Applying Lemma B.2 to equation (A.20) yields

$$\begin{aligned}
\|\nabla(h^{k+1}-h^k)\|_s^2 &\leq C(c\|\Delta(h^{k+1}-h^k)\|_{s-1}^2+\lambda c_1\|\nabla(h^{k+1}-h^k)\|_{s-1}^2) \\
&\leq \frac{C}{\lambda c_1}[\frac{1}{Re^2}\|D(\frac{1}{r})\|_{s_1+1}^2\|\Delta P(\mathbf{u}^k-\mathbf{u}^{k-1})\|_{s-2}^2 \\
&\quad +\|Q((\mathbf{q}+\nabla\phi)\cdot\nabla P(\mathbf{u}^k-\mathbf{u}^{k-1}))\|_{s-1}^2] \\
&\leq \frac{C}{\lambda c_1}[\frac{1}{Re^2}\|D(\frac{1}{r})\|_{s_1+1}^2\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_s^2+\|\mathbf{q}+\nabla\phi\|_{s-1}^2\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_s^2] \\
&\leq \frac{K_1}{\lambda c_1}\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_s^2 \qquad (A.21)
\end{aligned}$$

where $s_1 = max\{s-2,2\}$ and K_1 depends on $\|\mathbf{q}\|_{s-1,T}$, $\|f\|_{s-2,T}$, $\|r\|_{s,T}$, Re, and s. Next we will apply Lemma B.3 to equation (A.19) to derive an estimate for $\mathbf{u}^k$. We start by letting

$$\begin{aligned}
h &= r \\
a(h) &= \frac{p'(h)}{h} \\
b(h) &= \frac{1}{h} \\
\mathbf{F} &= Q[(\mathbf{q}+\nabla\phi)\cdot\nabla(P(\mathbf{u}^k-\mathbf{u}^{k-1}))]-Q[\frac{1}{rRe}\Delta P(\mathbf{u}^k-\mathbf{u}^{k-1})]
\end{aligned}$$

We will now show the regularity for each.

$$\|Q(\mathbf{q}+\nabla\phi)\cdot\nabla P(\mathbf{u}^k-\mathbf{u}^{k-1})\|_s^2 \leq C\|\mathbf{q}+\nabla\phi\|_s^2\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_s^2$$

and by Lemma B.1

$$\|Q(\frac{1}{rRe}\Delta P(\mathbf{u}^k-\mathbf{u}^{k-1}))\|_s^2 \leq C\|\frac{1}{rRe}\|_s^2\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_{s+1}^2$$

Applying Lemma B.3 yields

$$\begin{aligned}
&\|\mathbf{u}^{k+1}-\mathbf{u}^k\|_s^2+\frac{c_2}{Re}\int_0^t\|D(\mathbf{u}^{k+1}-\mathbf{u}^k)\|_s^2 d\tau\\
&\qquad\le C[\int_0^T\|Q[(\mathbf{q}+\nabla\phi)\cdot\nabla(P(\mathbf{u}^k-\mathbf{u}^{k-1}))]-Q[\frac{1}{rRe}\Delta P(\mathbf{u}^k-\mathbf{u}^{k-1})]\|_s^2 d\tau]\\
&\qquad\le CT[\|\mathbf{q}+\nabla\phi\|_{s,T}^2\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_{s,T}^2+\|\frac{1}{rRe}\|_{s,T}^2\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_{s,T}^2]\\
&\qquad\quad +C\|\frac{1}{rRe}\|_{s,T}^2\int_0^t\|D(\mathbf{u}^k-\mathbf{u}^{k-1})\|_s^2 d\tau\\
&\qquad\le CT(\|\mathbf{q}+\nabla\phi\|_{s,T}^2+\|\frac{1}{rRe}\|_{s,T}^2)\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_{s,T}^2\\
&\qquad\quad +\frac{C}{Re^2}\|\frac{1}{r}\|_{s,T}^2\int_0^T\|D(\mathbf{u}^k-\mathbf{u}^{k-1})\|_s^2 d\tau\\
&\qquad\le TK_2\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_{s,T}^2+\frac{1}{Re^2}K_2\int_0^T\|D(\mathbf{u}^k-\mathbf{u}^{k-1})\|_s^2 d\tau
\end{aligned}$$

where K_2 depends on $\|\mathbf{q}\|_{s,T}$, $\|f\|_{s-1,T}$, $\|r\|_{s,T}$, c_1, and s. For a small enough time interval $T\le\delta$ and $\frac{1}{Re}\le\delta$

$$\|\mathbf{u}^{k+1}-\mathbf{u}^k\|_{s,T}^2+\frac{c_2}{Re}\int_0^T\|D(\mathbf{u}^{k+1}-\mathbf{u}^k)\|_s^2 d\tau\le K_3[\|\mathbf{u}^k-\mathbf{u}^{k-1}\|_{s,T}^2+\frac{c_2}{Re}\int_0^T\|D(\mathbf{u}^k-\mathbf{u}^{k-1})\|_s^2 d\tau]$$

where $0<K_3<1$. It follows that $\|\mathbf{u}^{k+1}-\mathbf{u}^k\|_{s,T}^2+\frac{c_2}{Re}\int_0^T\|D(\mathbf{u}^{k+1}-\mathbf{u}^k)\|_s^2 d\tau\to 0$ as $k\to\infty$. Inequality (A.21) implies that $\|\nabla(h^{k+1}-h^k)\|_{s,T}^2\to 0$ as $k\to\infty$. Then Lemma B.1 implies $\|h^{k+1}-h^k\|_{s+1,T}^2\to 0$ as $k\to\infty$. Therefore, there exist $h\in C([0,T],C^5(\Omega))\cap L^\infty([0,T],H^{s+1}(\Omega))$ and $\mathbf{u}\in C([0,T],C^4(\Omega))\cap L^\infty([0,T],H^s(\Omega))\cap L^2([0,T],H^{s+1}(\Omega))$ such that
$\|\mathbf{u}^k-\mathbf{u}\|_{s,T}^2+\frac{c_2}{Re}\int_0^T\|D(\mathbf{u}^k-\mathbf{u})\|_s^2 d\tau\to 0$ as $k\to\infty$ and $\|h^k-h\|_{s+1,T}^2\to 0$ as $k\to\infty$. Since h^{k+1}, $\mathbf{u}^{k+1}$ is a solution of (A.9), (A.10) for $k\ge 0$, it follows that h, $\mathbf{u}$ is a solution of (A.7), (A.8).

APPENDIX B: SUPPORTING LEMMAS

Lemma B.1 Standard Sobolev Space Inequalities

(a) If $f \in H^n(\Omega)$, where $\Omega \subset \mathbb{R}^N$, and $r = \beta m + (1-\beta)n$, with $0 \le \beta \le 1$ and $m \le n$, then

$$\|f\|_r \le C\|f\|_m^{\beta}\|f\|_n^{1-\beta}$$

Here C is a constant which depends on m, n, N, and Ω.

(b) Let $g(u)$ be a smooth function on G, where $u(\mathbf{x})$ is a continuous function and where $u(\mathbf{x}) \in G_1$ for $\mathbf{x} \in \Omega$ and $G_1 \subset G$ and $u \in H^r(\Omega) \cap L^\infty(\Omega)$. Then for $r \ge 1$,

$$\|D^r(g(u))\|_0 \le C|\frac{dg}{du}|_{r-1,\bar{G}_1}(1+|u|_{L^\infty})^{r-1}\|Du\|_{r-1}$$

where $|h|_{r,\bar{G}_1} = max\{|\frac{d^jh}{du^j}(u_*)| : u_* \in \bar{G}_1, 0 \le j \le r\}$, and where C depends on r and Ω.
(c) And,

$$\|g(u)-g(v)\|_r \le C|\frac{dg}{du}|_{r,\bar{G}_1}(1+|u|_{L^\infty}+|v|_{L^\infty})(\|u\|_r+\|v\|_r)\|u-v\|_r$$

where $|\frac{dg}{du}|_{r,\bar{G}_1} = max\{|\frac{d^{j+1}g}{du^{j+1}}(u_*)| : u_* \in \bar{G}_1, 0 \le j \le r\}$, where $\Omega = \mathbb{T}^N$, the N-dimensional torus, and where the constant C depends on r, Ω.

(d) If $f \in H^{s_1}(\Omega)$, $g \in H^{s_2}(\Omega)$, and $s_3 = min\{s_1, s_2, s_1+s_2-s_0\} \ge 0$, where $s_0 = [\frac{N}{2}]+1$, then $fg \in H^{s_3}(\Omega)$, and $\|fg\|_{s_3} \le C\|f\|_{s_1}\|g\|_{s_2}$ where the constant C depends on s_1, s_2, Ω. We note that $s_0 = 2$ for $N = 2$ or $N = 3$.

(e) If $Df \in H^{r_1}(\Omega)$, $g \in H^{r-1}(\Omega)$, where $r_1 = max\{r-1, s_0\}$, $s_0 = [\frac{N}{2}]+1$, then for any $r \ge 1$, f, g satisfy the estimate $\|D^\alpha(fg) - fD^\alpha g\|_0 \le C\|Df\|_{r_1}\|g\|_{r-1}$, where $r = |\alpha|$, and the constant C depends on r, Ω.

(f) Let $\mathbf{u}$*,* $\mathbf{w} \in H^2(\Omega)$ *be functions on a bounded, open, convex domain* Ω*. If* $\mathbf{u}(\mathbf{x}_0) = \mathbf{w}(\mathbf{x}_0)$ *at a single point* $\mathbf{x}_0 \in \Omega$*, then* $\mathbf{u}-\mathbf{w}$ *and* $\mathbf{u}$ *satisfy the estimates*

$$\|\mathbf{u}-\mathbf{w}\|_0^2 \leq C\|\nabla(\mathbf{u}-\mathbf{w})\|_1^2$$
$$\|\mathbf{u}\|_0^2 \leq C\|\mathbf{w}\|_0^2 + C\|\nabla\mathbf{w}\|_1^2 + C\|\nabla\mathbf{u}\|_1^2$$
$$|\mathbf{u}-\mathbf{w}|_{L^\infty} \leq C\|\nabla(\mathbf{u}-\mathbf{w})\|_1^2$$

(g) Let $\eta(\cdot)$ *be a nonnegative, absolutely continuous functions on* $[0,T]$*, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t)$$

where $\phi(t)$ *and* $\psi(t)$ *are nonnegative, summable functions on* $[0,T]$*. Then*

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}[\eta(0) + \int_0^t \psi(s)ds]$$

for all $0 \leq t \leq T$*.*

(h) Given $a \in H^{r+1}(\Omega)$*,* $f \in H^r(\Omega)$*,* $r > \frac{N}{2}+3$*,* $\Omega = \mathbb{T}^N$*, then* $P(a\nabla f) \in H^r(\Omega)$*, and* $\|P(a\nabla f)\|_r \leq C\|Da\|_r\|\nabla f\|_{r-1}$*, where P is the projection onto the solenoidal vector field.*

(i) If $r > \frac{N}{2}+1$ *and* $\Omega = \mathbb{T}^N$*, then* $\|Q(a\Delta P\mathbf{u})\|_{r-1} \leq C\|Da\|_{r-2}\|P\mathbf{u}\|_r$*. Also,* $\|Q(\mathbf{v}\cdot\nabla P\mathbf{u})\|_r \leq C\|\mathbf{v}\|_r\|\mathbf{u}\|_r$*.*

(j) If a is a sufficiently smooth function of u and $u \in H^s(\Omega)$ *and* u_0 *is a constant such that* $|u-u_0|_{L^\infty} \leq R$*, then*

$$\|D(a(u))\|_{s-1}^2 \leq C|\frac{da}{du}|_{s_1,\bar{G}_1}^2\|\nabla u\|_{s-1}^2$$

where $|\frac{da}{du}|_{s_1,\bar{G}_1} = max\{|\frac{d^{j+1}a}{du^{j+1}}(u_*)| : u_* \in \bar{G}_1, 0 \leq j \leq s_1\}$*, where* $s_1 = max\{s-1,2\}$*, and where* $\Omega = \mathbb{T}^N$*, the N-dimensional torus, and C depends on s, R,* u_0*.*

(k) If f is sufficiently smooth and $\int_\Omega f d\mathbf{x} = 0$ where $\Omega = \mathbb{T}^2$, the two-dimensional torus, then there exist a unique solution $\nabla\phi$ to $\Delta\phi = f$ and $\|\nabla\phi\|_r \leq C\|f\|_{r-1}$ where C depends on r.

The proofs of (a)-(k) can be found in [5], [11], and [12].

Lemma B.2: $\|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2$ Estimate

If $\mathbf{v}$, $\mathbf{w}$, $a(h)$, $b(h)$, and $\mathbf{F}$ are sufficiently smooth in

$$c\Delta^2\rho - \nabla\cdot\lambda(a(h)\nabla\rho) = \nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{w}) - \nabla\cdot\frac{1}{Re}(b(h)\Delta\mathbf{w}) + \nabla\cdot\mathbf{F} \tag{B.1}$$

where a is a positive smooth function of h, $0 < c_1 < a(h)$, $\nabla\cdot\mathbf{w} = 0$, c is a positive constant, and $\|Da(h)\|_{r_1} \leq \varepsilon c_1$ where $r_1 = max\{r-1, 2\}$, then

$$\|\nabla\rho\|_{r+1}^2 \leq C(c\|\Delta\rho\|_r^2 + \lambda c_1\|\nabla\rho\|_r^2) \leq \frac{C}{\lambda c_1}[\frac{1}{Re^2}\|Db(h)\|_{r_1+1}^2\|\Delta\mathbf{w}\|_{r-1}^2 + \|\mathbf{v}\cdot\nabla\mathbf{w}\|_r^2 + \|\mathbf{F}\|_r^2]$$

where C depends on r, and $r \geq 2$.

Proof:

Let $\bar{\rho} = \rho - \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2}\rho d\mathbf{x}$. Start by taking the inner product of equation (B.1) against $\bar{\rho}$ to obtain

$$\begin{aligned}(c\Delta^2\rho, \bar{\rho}) - (\nabla\cdot\lambda(a(h)\nabla\rho), \bar{\rho}) &= (\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{w}) - \nabla\cdot\frac{1}{Re}(b(h)\Delta\mathbf{w}) + \nabla\cdot\mathbf{F}, \bar{\rho}) \\ &= (\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{w}), \bar{\rho}) - (\nabla\cdot\frac{1}{Re}(b(h)\Delta\mathbf{w}), \bar{\rho}) + (\nabla\cdot\mathbf{F}, \bar{\rho})\end{aligned}$$

Via integration by parts we obtain

$$(c\Delta\rho, \Delta\rho) + \lambda(a(h)\nabla\rho, \nabla\rho) = -(\mathbf{v}\cdot\nabla\mathbf{w}, \nabla\rho) - \frac{1}{Re}(\nabla b(h)\cdot\Delta\mathbf{w}, \bar{\rho}) - (\mathbf{F}, \nabla\rho)$$

from this equation and $0 < c_1 < a(h)$ so we can say

$$c\|\Delta\rho\|_0^2 + \lambda c_1\|\nabla\rho\|_0^2 \leq [|(\mathbf{v}\cdot\nabla\mathbf{w}, \nabla\rho)| + |\frac{1}{Re}(\nabla b(h)\cdot\Delta\mathbf{w}, \bar{\rho})| + |(\mathbf{F}, \nabla\rho)|] \tag{B.2}$$

Using Cauchy's inequality with ε, the first term in inequality (B.2) becomes

$$|(\mathbf{v}\cdot\nabla\mathbf{w}, \nabla\rho)| \leq \frac{1}{4\varepsilon\lambda c_1}\|\mathbf{v}\cdot\nabla\mathbf{w}\|_0^2 + \varepsilon\lambda c_1\|\nabla\rho\|_0^2 \tag{B.3}$$

In the same way the second term becomes

$$\frac{1}{Re}|(\nabla b(h)\cdot\Delta\mathbf{w}),\bar{\rho})| \leq \frac{1}{4\varepsilon Re^2\lambda c_1}\|\nabla b(h)\cdot\Delta\mathbf{w}\|_0^2+\varepsilon\lambda c_1\|\bar{\rho}\|_0^2$$
$$\leq \frac{1}{4\varepsilon Re^2\lambda c_1}|\nabla b(h)|_{L^\infty}^2\|\Delta\mathbf{w}\|_0^2+\varepsilon\lambda c_1\|\bar{\rho}\|_0^2 \tag{B.4}$$

Similarly, the last term in inequality (B.2) becomes

$$|(\mathbf{F},\nabla\rho)| \leq \frac{1}{4\varepsilon\lambda c_1}\|\mathbf{F}\|_0^2+\varepsilon\lambda c_1\|\nabla\rho\|_0^2 \tag{B.5}$$

Combining and simplifying inequalities (B.2), (B.3), (B.4), and (B.5) we obtain

$$c\|\Delta\rho\|_0^2+\lambda c_1\|\nabla\rho\|_0^2 \leq [\frac{1}{4\varepsilon\lambda c_1}\|(\mathbf{v}\cdot\nabla\mathbf{w})\|_0^2+\frac{1}{4\varepsilon\lambda c_1}\frac{1}{Re^2}|\nabla b(h)|_{L^\infty}^2\|\Delta\mathbf{w}\|_0^2$$
$$+\varepsilon\lambda c_1\|\bar{\rho}\|_0^2+\frac{1}{4\varepsilon\lambda c_1}\|\mathbf{F}\|_0^2+2\varepsilon\lambda c_1\|\nabla\rho\|_0^2] \tag{B.6}$$

Using Poincaré's inequality

$$\|\bar{\rho}\|_0^2 \leq C\|\nabla\rho\|_0^2$$

we derive from inequality (B.6) where $\varepsilon \leq (\frac{1}{2C+4})$.

$$c\|\Delta\rho\|_0^2+\lambda c_1\|\nabla\rho\|_0^2 \leq \frac{C}{\lambda c_1}[\|(\mathbf{v}\cdot\nabla\mathbf{w})\|_0^2+\frac{1}{Re^2}|\nabla b(h)|_{L^\infty}^2\|\Delta\mathbf{w}\|_0^2+\|\mathbf{F}\|_0^2] \tag{B.7}$$

Applying D^α to equation (B.1) gives

$$c\Delta^2 D^\alpha\rho-\nabla\cdot\lambda(a(h)D^\alpha(\nabla\rho)) = D^\alpha(\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{w}))-\nabla\cdot(\frac{1}{Re}b(h)D^\alpha(\Delta\mathbf{w}))+D^\alpha(\mathbf{G}) \tag{B.8}$$

where

$$D^\alpha(\mathbf{G}) = -\frac{1}{Re}[D^\alpha(\nabla\cdot(b(h)\Delta\mathbf{w}))-\nabla\cdot(b(h)D^\alpha(\Delta\mathbf{w}))]$$
$$+\nabla\cdot[\lambda D^\alpha(a(h)\nabla\rho)-\lambda(a(h)D^\alpha(\nabla\rho))]+D^\alpha(\nabla\cdot\mathbf{F})$$

For notational purposes, we will write

$$D^\alpha\rho = \rho_\alpha$$

$$D^\alpha\mathbf{w} = \mathbf{w}_\alpha$$

$$D^\alpha\mathbf{F} = \mathbf{F}_\alpha$$

So now the equation (B.8) can be written as

$$c\Delta^2\rho_\alpha - \nabla\cdot\lambda(a(h)\nabla\rho_\alpha) = \nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{w})_\alpha - \nabla\cdot\frac{1}{Re}(b(h)\Delta\mathbf{w}_\alpha) + \mathbf{G}_\alpha$$

Taking the inner product of both sides of this equation with respect to ρ_α gives

$$\begin{aligned}(c\Delta^2\rho_\alpha,\rho_\alpha) &- (\nabla\cdot\lambda(a(h)\nabla\rho_\alpha),\rho_\alpha)\\ &= (\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{w})_\alpha,\rho_\alpha) - (\nabla\cdot\frac{1}{Re}(b(h)\Delta\mathbf{w}_\alpha),\rho_\alpha) + (\mathbf{G}_\alpha,\rho_\alpha)\end{aligned}$$

Via integration by parts we obtain

$$(c\Delta\rho_\alpha,\Delta\rho_\alpha) + (\lambda a(h)\nabla\rho_\alpha,\nabla\rho_\alpha) = -((\mathbf{v}\cdot\nabla\mathbf{w})_\alpha,\nabla\rho_\alpha) - \frac{1}{Re}(\nabla b(h)\cdot\Delta\mathbf{w}_\alpha,\rho_\alpha) + (\mathbf{G}_\alpha,\rho_\alpha)$$

From this equation and $0 < c_1 < a(h)$ we can say

$$c\|\Delta\rho_\alpha\|_0^2 + \lambda c_1\|\nabla\rho_\alpha\|_0^2 \le [|((\mathbf{v}\cdot\nabla\mathbf{w})_\alpha,\nabla\rho_\alpha)| + \frac{1}{Re}|(\nabla b(h)\cdot\Delta\mathbf{w}_\alpha,\rho_\alpha)| + |(\mathbf{G}_\alpha,\rho_\alpha)|] \tag{B.9}$$

We can estimate the first and second term in inequality (B.9) by applying Cauchy's inequality with ε. The first term is as follows

$$\begin{aligned}|((\mathbf{v}\cdot\nabla\mathbf{w})_\alpha,\nabla\rho_\alpha)| &\le \|(\mathbf{v}\cdot\nabla\mathbf{w})_\alpha\|_0\|\nabla\rho_\alpha\|_0\\ &\le \frac{1}{4\varepsilon\lambda c_1}\|(\mathbf{v}\cdot\nabla\mathbf{w})_\alpha\|_0^2 + \varepsilon\lambda c_1\|\nabla\rho_\alpha\|_0^2\end{aligned} \tag{B.10}$$

The second term in inequality (B.9) is estimated as follows

$$\begin{aligned}\frac{1}{Re}|(\nabla b(h)\cdot\Delta\mathbf{w}_\alpha,\rho_\alpha)| &\le \frac{1}{Re}|(\nabla b(h)\cdot\Delta\mathbf{w}_{\alpha-\beta},\rho_{\alpha+\beta})| + \frac{1}{Re}|(\nabla b(h)_\beta\cdot\Delta\mathbf{w}_{\alpha-\beta},\rho_\alpha)|\\ &\le \frac{1}{Re}\|\nabla b(h)\cdot\Delta\mathbf{w}_{\alpha-\beta}\|_0\|\rho_{\alpha+\beta}\|_0 + \frac{1}{Re}\|\nabla b(h)_\beta\cdot\Delta\mathbf{w}_{\alpha-\beta}\|_0\|\rho_\alpha\|_0\\ &\le \frac{1}{Re}|Db(h)|_{L^\infty}\|\Delta\mathbf{w}_{\alpha-\beta}\|_0\|\rho_{\alpha+\beta}\|_0 + \frac{1}{Re}|Db(h)_\beta|_{L^\infty}\|\Delta\mathbf{w}_{\alpha-\beta}\|_0\|\rho_\alpha\|_0\\ &\le \frac{1}{4\varepsilon Re^2\lambda c_1}|Db(h)|_{L^\infty}^2\|\Delta\mathbf{w}_{\alpha-\beta}\|_0^2 + \frac{1}{4\varepsilon Re^2\lambda c_1}|Db(h)_\beta|_{L^\infty}^2\|\Delta\mathbf{w}_{\alpha-\beta}\|_0^2\\ &\quad + \varepsilon\lambda c_1\|\rho_{\alpha+\beta}\|_0^2 + \varepsilon\lambda c_1\|\rho_\alpha\|_0^2\end{aligned} \tag{B.11}$$

where $|\beta| = 1$ and $1 \leq |\alpha|$. The last term in inequality (B.9) is estimated as follows. Start by integrating by parts to obtain

$$\begin{aligned}
|(\mathbf{G}_\alpha, \rho_\alpha)| &\leq \lambda |(\nabla \cdot [(a(h)\nabla\rho)_\alpha - a(h)\nabla\rho_\alpha], \rho_\alpha)| \\
&\quad + \frac{1}{Re}|(\nabla \cdot [(b(h)\Delta\mathbf{w})_\alpha - b(h)\Delta\mathbf{w}_\alpha], \rho_\alpha)| + |(\nabla \cdot \mathbf{F}_\alpha, \rho_\alpha)| \\
&\leq \lambda |(([(a(h)\nabla\rho)_\alpha - a(h)\nabla\rho_\alpha], \nabla\rho_\alpha)| \\
&\quad + \frac{1}{Re}|(([(b(h)\Delta\mathbf{w})_\alpha - b(h)\Delta\mathbf{w}_\alpha], \nabla\rho_\alpha)| + |(\mathbf{F}_\alpha, \nabla\rho_\alpha)|
\end{aligned}$$

From this we obtain the following inequality

$$\begin{aligned}
|(\mathbf{G}_\alpha, \rho_\alpha)| &\leq \lambda \|(a(h)\nabla\rho)_\alpha - a(h)\nabla\rho_\alpha\|_0 \|\nabla\rho_\alpha\|_0 \\
&\quad + \frac{1}{Re}\|(b(h)\Delta\mathbf{w})_\alpha - b(h)\Delta\mathbf{w}_\alpha\|_0 \|\nabla\rho_\alpha\|_0 + \|\mathbf{F}_\alpha\|_0 \|\nabla\rho_\alpha\|_0
\end{aligned} \tag{B.12}$$

Applying the commutator estimate from Lemma B.1, the first and second part of inequality (B.12) becomes

$$\begin{aligned}
&\|(a(h)\nabla\rho)_\alpha - a(h)\nabla\rho_\alpha\|_0 \leq C\|Da(h)\|_{k_1}\|\nabla\rho\|_{k-1} \\
&\|(b(h)\Delta\mathbf{w})_\alpha - b(h)\Delta\mathbf{w}_\alpha)\|_0 \leq C\|Db(h)\|_{k_1}\|\Delta\mathbf{w}\|_{k-1}
\end{aligned} \tag{B.13}$$

where $k_1 = max\{k-1, 2\}$ and $k = |\alpha|$.

Combining inequalities (B.13) with (B.12) gives

$$\begin{aligned}
|(\mathbf{G}_\alpha, \rho_\alpha)| &\leq C\lambda \|Da(h)\|_{k_1}\|\nabla\rho\|_{k-1}\|\nabla\rho_\alpha\|_0 \\
&\quad + \frac{C}{Re}\|Db(h)\|_{k_1}\|\Delta\mathbf{w}\|_{k-1}\|\nabla\rho_\alpha\|_0 + \|\mathbf{F}_\alpha\|_0\|\nabla\rho_\alpha\|_0
\end{aligned} \tag{B.14}$$

Now applying Cauchy's inequality with ε to (B.14) gives

$$\begin{aligned}
|(\mathbf{G}_\alpha, \rho_\alpha)| &\leq C\lambda \|Da(h)\|_{k_1}\|\nabla\rho\|_{k-1}\|\nabla\rho_\alpha\|_0 + \frac{C}{4\varepsilon Re^2\lambda c_1}\|Db(h)\|_{k_1}^2\|\Delta\mathbf{w}\|_{k-1}^2 \\
&\quad + \frac{1}{4\varepsilon\lambda c_1}\|\mathbf{F}_\alpha\|_0^2 + C\varepsilon\lambda c_1\|\nabla\rho_\alpha\|_0^2
\end{aligned} \tag{B.15}$$

Combining inequalities (B.9), (B.10), (B.11), and (B.15) we obtain

$$\begin{aligned}
&c\|\Delta\rho_\alpha\|_0^2 + c_1\lambda\|\nabla\rho_\alpha\|_0^2 \\
&\qquad \leq [\frac{1}{4\varepsilon\lambda c_1}\|(\mathbf{v}\cdot\nabla\mathbf{w})_\alpha\|_0^2 + \frac{1}{4\varepsilon\lambda c_1}(\frac{1}{Re^2}|Db(h)|_{L^\infty}^2 + \frac{1}{Re^2}|Db(h)_\beta|_{L^\infty}^2)\|\Delta\mathbf{w}_{\alpha-\beta}\|_0^2 \\
&\qquad\quad + C\lambda\|Da(h)\|_{k_1}\|\nabla\rho\|_{k-1}\|\nabla\rho_\alpha\|_0 + \frac{C}{4\varepsilon}\frac{1}{Re^2\lambda c_1}\|Db(h)\|_{k_1}^2\|\Delta\mathbf{w}\|_{k-1}^2 \\
&\qquad\quad + \frac{1}{4\varepsilon\lambda c_1}\|\mathbf{F}_\alpha\|_0^2 + C\varepsilon\lambda c_1\|\nabla\rho_\alpha\|_0^2 + \varepsilon\lambda c_1\|\rho_{\alpha+\beta}\|_0^2 + \varepsilon\lambda c_1\|\rho_\alpha\|_0^2]
\end{aligned} \tag{B.16}$$

Summing inequality (B.16) over $1 \leq |\alpha| \leq r$ and over $|\beta| = 1$ as well as adding the L^2 estimate inequality (B.7),we get

$$\begin{aligned}
c\|\Delta\rho\|_r^2 + c_1\lambda\|\nabla\rho\|_r^2 \leq C[&\frac{1}{4\varepsilon\lambda c_1}\|\mathbf{v}\cdot\nabla\mathbf{w}\|_r^2 + \frac{1}{4\varepsilon\lambda c_1}(\frac{1}{Re^2}|Db(h)|_{L^\infty}^2 + \frac{1}{Re^2}|D^2b(h)|_{L^\infty}^2)\|\Delta\mathbf{w}\|_{r-1}^2 \\
&+ \lambda\|Da(h)\|_{r_1}\|\nabla\rho\|_r^2 + \frac{1}{4\varepsilon Re^2\lambda c_1}\|Db(h)\|_{r_1}^2\|\Delta\mathbf{w}\|_{r-1}^2 \\
&+ \frac{1}{4\varepsilon\lambda c_1}\|\mathbf{F}\|_r^2] + C\varepsilon\lambda c_1\|\nabla\rho\|_r^2 + C\varepsilon\lambda c_1\|\nabla\rho\|_{r-1}^2
\end{aligned} \tag{B.17}$$

where C depends on r.

We are given that

$$\|Da(h)\|_{r_1} \leq \varepsilon c_1$$

From Lemma B.1 we have:

$$|Db(h)|_{L^\infty}^2 + |D^2b(h)|_{L^\infty}^2 + \|Db(h)\|_{r_1}^2 \leq C\|Db(h)\|_{r_1+1}^2$$

Combining like terms in inequality (B.17) gives us

$$c\|\Delta\rho\|_r^2 + c_1\lambda\|\nabla\rho\|_r^2 \leq \frac{C}{4\varepsilon\lambda c_1}[\frac{1}{Re^2}\|Db(h)\|_{r_1+1}^2\|\Delta\mathbf{w}\|_{r-1}^2 + \|\mathbf{v}\cdot\nabla\mathbf{w}\|_r^2 + \|\mathbf{F}\|_r^2] + C\varepsilon\lambda c_1\|\nabla\rho\|_r^2$$

where $\varepsilon \leq \frac{1}{2C}$ and where C depends on r. Combining like terms one more time gives

$$\|\nabla\rho\|_{r+1}^2 \leq C(c\|\Delta\rho\|_r^2 + \lambda c_1\|\nabla\rho\|_r^2) \leq \frac{C}{\lambda c_1}[\frac{1}{Re^2}\|Db(h)\|_{r_1+1}^2\|\Delta\mathbf{w}\|_{r-1}^2 + \|\mathbf{v}\cdot\nabla\mathbf{w}\|_r^2 + \|\mathbf{F}\|_r^2] \tag{B.18}$$

where C depends on r which completes the proof.

Lemma B.3: $\|\mathbf{w}\|_r^2 + \frac{c_2}{Re}\int_0^t \|\nabla \mathbf{w}\|_r^2 d\tau$ Estimate

If $\mathbf{v} \in H^r(\Omega)$, $a(h) \in H^{r+1}(\Omega)$, $b(h) \in H^{r+1}(\Omega)$, $f \in H^r(\Omega)$, *and* $\mathbf{F} \in H^r(\Omega)$, $\Omega = \mathbb{T}^2$, $\|D(a(h))\|_{r_1} \leq \varepsilon c_1$, *in*

$$\mathbf{w}_t + \mathbf{v}\cdot\nabla\mathbf{w} = \frac{1}{Re}b(h)\Delta\mathbf{w} - \lambda a(h)\nabla\rho + c\nabla\Delta\rho + \mathbf{F} \tag{B.19}$$

where $\mathbf{w}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x})$, $\mathbf{w}_0 \in H^r(\mathbb{T}^2)$, $\nabla\cdot\mathbf{w} = 0$, $\nabla\cdot\mathbf{v} = f$, $b(h) > c_2 > 0$, $a(h) > c_1 > 0$, *and* $t \in [0,T]$ *then*

$$\|\mathbf{w}\|_r^2 + \frac{c_2}{Re}\int_0^T \|D\mathbf{w}\|_r^2 d\tau \leq C[\|\mathbf{w}_0\|_r^2 + \int_0^T \|\mathbf{F}\|_r^2 d\tau]$$

where C *depends on* r, *and* $r \geq 3$.

Proof:

We will start by determining an L^2 estimate for $\mathbf{w}$. Let $\bar{\rho} = \rho - \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2}\rho d\mathbf{x}$. We start by taking the inner product of (B.19) against $\mathbf{w}$

$$(\mathbf{w}_t,\mathbf{w}) = -(\mathbf{v}\cdot\nabla\mathbf{w},\mathbf{w}) + \frac{1}{Re}(b(h)\Delta\mathbf{w},\mathbf{w}) - \lambda(a(h)\nabla\rho,\mathbf{w}) + (c\nabla\Delta\rho,\mathbf{w}) + (\mathbf{F},\mathbf{w}) \tag{B.20}$$

The term on the left becomes

$$(\mathbf{w}_t,\mathbf{w}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_0^2 \tag{B.21}$$

Because $\nabla\cdot\mathbf{v} = f$ the first term on the right becomes

$$\begin{aligned} -(\mathbf{v}\cdot\nabla\mathbf{w},\mathbf{w}) &\leq |(\mathbf{v}\cdot\nabla\mathbf{w},\mathbf{w})| \\ &\leq \frac{1}{2}|\nabla\cdot\mathbf{v}|_{L^\infty}\|\mathbf{w}\|_0^2 \\ &= \frac{1}{2}|f|_{L^\infty}\|\mathbf{w}\|_0^2 \end{aligned} \tag{B.22}$$

The second term on the right is

$$\frac{1}{Re}(b(h)\Delta\mathbf{w},\mathbf{w}) = \frac{1}{Re}\int_{\mathbb{T}^2} b(h)\Delta\mathbf{w}\cdot\mathbf{w}d\mathbf{x}$$

Via integration by parts we obtain

$$\frac{1}{Re}\int_{\mathbb{T}^2} b(h)\Delta\mathbf{w}\cdot\mathbf{w}d\mathbf{x} = -\frac{1}{Re}\int_{\mathbb{T}^2} b(h)|D\mathbf{w}|^2 d\mathbf{x} - \frac{1}{Re}\int_{\mathbb{T}^2}((\nabla b(h))^T\cdot\nabla\mathbf{w})\cdot\mathbf{w}d\mathbf{x}$$

Therefore, we have

$$\frac{1}{Re}(b(h)\Delta\mathbf{w},\mathbf{w}) \leq -\frac{c_2}{Re}\|D\mathbf{w}\|_0^2 + \frac{C}{Re}|Db(h)|_{L^\infty}\|D\mathbf{w}\|_0\|\mathbf{w}\|_0 \tag{B.23}$$

where $c_2 \leq b(h)$. The third term of the right side of (B.20) becomes

$$-\lambda(a(h)\nabla\rho,\mathbf{w}) = -\lambda\int_{\mathbb{T}^2} a(h)\nabla\rho\cdot\mathbf{w}d\mathbf{x} = -\lambda\int_{\mathbb{T}^2}\nabla\rho\cdot(a(h)\mathbf{w})d\mathbf{x}$$

Via integration by parts we obtain

$$\begin{aligned}-\lambda(a(h)\nabla\rho,\mathbf{w}) &= \lambda\int_{\mathbb{T}^2}\bar{\rho}\nabla\cdot(a(h)\mathbf{w})d\mathbf{x}\\ &= \lambda\int_{\mathbb{T}^2}(\bar{\rho}\nabla a(h))\cdot\mathbf{w}d\mathbf{x} + \lambda\int_{\mathbb{T}^2} a(h)\bar{\rho}(\nabla\cdot\mathbf{w})d\mathbf{x}\end{aligned}$$

Since $\nabla\cdot\mathbf{w} = 0$ we get

$$-\lambda(a(h)\nabla\rho,\mathbf{w}) = \lambda(\bar{\rho}\nabla a(h),\mathbf{w})$$

and by Poincaré's inequality

$$\begin{aligned}-\lambda(a(h)\nabla\rho,\mathbf{w}) &\leq \lambda|(a(h)\nabla\rho,\mathbf{w})|\\ &\leq C\lambda\|\bar{\rho}\|_0|Da(h)|_{L^\infty}\|\mathbf{w}\|_0\\ &\leq C\lambda\|\nabla\rho\|_0|Da(h)|_{L^\infty}\|\mathbf{w}\|_0\end{aligned} \tag{B.24}$$

Since $\nabla\cdot\mathbf{w} = 0$, the fourth term in (B.20) becomes

$$(c\nabla\Delta\rho,\mathbf{w}) = -(c\Delta\rho,\nabla\cdot\mathbf{w}) = 0 \tag{B.25}$$

The final term in (B.20) is bounded by

$$\begin{aligned}(\mathbf{F},\mathbf{w}) &\leq |(\mathbf{F},\mathbf{w})|\\ &\leq \|\mathbf{F}\|_0\|\mathbf{w}\|_0\end{aligned} \tag{B.26}$$

Combining inequalities, applying Cauchy's inequality with ε, and applying the Sobolev inequalities from Lemma B.1 to (B.20), (B.21), (B.22), (B.23), (B.24), (B.25), and (B.26) gives

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}\|_0^2+\frac{2c_2}{Re}\|\nabla\mathbf{w}\|_0^2 &\le |f|_{L^\infty}\|\mathbf{w}\|_0^2+\frac{C}{Re}|Db(h)|_{L^\infty}\|D\mathbf{w}\|_0\|\mathbf{w}\|_0\\ &\quad +C\lambda\|\nabla\rho\|_0|Da(h)|_{L^\infty}\|\mathbf{w}\|_0+2\|\mathbf{F}\|_0\|\mathbf{w}\|_0\\ &\le |f|_{L^\infty}\|\mathbf{w}\|_0^2+\frac{2\varepsilon c_2}{Re}\|D\mathbf{w}\|_0^2+\varepsilon\|\nabla\rho\|_0^2+\varepsilon\|\mathbf{F}\|_0^2\\ &\quad +\frac{C}{\varepsilon}(1+\frac{1}{c_2Re}\|Db(h)\|_{r_1}^2+\lambda^2\|Da(h)\|_{r_1}^2)\|\mathbf{w}\|_0^2\end{aligned}$$

where $r_1=max\{r-1,2\}$ and $0<\varepsilon<1$. Rearranging terms yields

$$\begin{aligned}&\frac{d}{dt}\|\mathbf{w}\|_0^2+\frac{2c_2(1-\varepsilon)}{Re}\|\nabla\mathbf{w}\|_0^2\\ &\quad\le \varepsilon\|\nabla\rho\|_0^2+\varepsilon\|\mathbf{F}\|_0^2+(|f|_{L^\infty}+\frac{C}{\varepsilon}(1+\frac{1}{c_2Re}\|Db(h)\|_{r_1}^2+\lambda^2\|Da(h)\|_{r_1}^2))\|\mathbf{w}\|_0^2\end{aligned}\tag{B.27}$$

where C depends on r.

We now apply the D^α operator to (B.19) and obtain

$$(D^\alpha\mathbf{w})_t=-\mathbf{v}\cdot\nabla(D^\alpha\mathbf{w})+\frac{1}{Re}b(h)\Delta(D^\alpha\mathbf{w})-\lambda a(h)\nabla(D^\alpha\rho)+D^\alpha(c\nabla\Delta\rho)+\mathbf{H}_\alpha\tag{B.28}$$

where

$$\begin{aligned}\mathbf{H}_\alpha=\frac{1}{Re}&[D^\alpha(b(h)\Delta\mathbf{w})-b(h)\Delta(D^\alpha\mathbf{w})]-\lambda[D^\alpha(a(h)\nabla\rho)-a(h)\nabla(D^\alpha\rho)]\\ &-[D^\alpha(\mathbf{v}\cdot\nabla\mathbf{w})-\mathbf{v}\cdot\nabla(D^\alpha\mathbf{w})]+D^\alpha\mathbf{F}\end{aligned}$$

For notational purposes, we will write

$$D^\alpha\rho=\rho_\alpha$$

$$D^\alpha\mathbf{w}=\mathbf{w}_\alpha$$

$$D^\alpha\mathbf{F}=\mathbf{F}_\alpha$$

We now take the inner product of both sides of equation (B.28) with $\mathbf{w}_\alpha$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}_\alpha\|_0^2&=((\mathbf{w}_\alpha)_t,\mathbf{w}_\alpha)\\ &=-(\mathbf{v}\cdot\nabla\mathbf{w}_\alpha,\mathbf{w}_\alpha)+\frac{1}{Re}(b(h)\Delta\mathbf{w}_\alpha,\mathbf{w}_\alpha)-\lambda(a(h)\nabla\rho_\alpha,\mathbf{w}_\alpha)\\ &\quad+(c\nabla\Delta\rho_\alpha,\mathbf{w}_\alpha)+(\mathbf{H}_\alpha,\mathbf{w}_\alpha)\end{aligned}\tag{B.29}$$

We will now estimate each term on the right side of equation (B.29). Because $\nabla \cdot \mathbf{v} = f$ the first term on the right becomes

$$\begin{aligned} -(\mathbf{v}\cdot\nabla\mathbf{w}_\alpha, \mathbf{w}_\alpha) &\leq |(\mathbf{v}\cdot\nabla\mathbf{w}_\alpha, \mathbf{w}_\alpha)| \\ &\leq \frac{1}{2}|\nabla\cdot\mathbf{v}|_{L^\infty}\|\mathbf{w}_\alpha\|_0^2 \\ &= \frac{1}{2}|f|_{L^\infty}\|\mathbf{w}_\alpha\|_0^2 \end{aligned} \tag{B.30}$$

Via integration by parts the second term in inequality (B.29) becomes

$$\begin{aligned} \frac{1}{Re}(b(h)\Delta\mathbf{w}_\alpha, \mathbf{w}_\alpha) &= \frac{1}{Re}\int_{\mathbb{T}^2} b(h)\Delta\mathbf{w}_\alpha\cdot\mathbf{w}_\alpha d\mathbf{x} \\ &= -\frac{1}{Re}\int_{\mathbb{T}^2} b(h)|D\mathbf{w}_\alpha|^2 d\mathbf{x} - \frac{1}{Re}\int_{\mathbb{T}^2}((\nabla b(h))^T\cdot\nabla\mathbf{w}_\alpha)\cdot\mathbf{w}_\alpha d\mathbf{x} \end{aligned}$$

We can use Cauchy's inequality with ε, Sobolev inequalities and $c_2 \leq b(h)$ to obtain

$$\begin{aligned} \frac{1}{Re}(b(h)\Delta\mathbf{w}_\alpha, \mathbf{w}_\alpha) &\leq -\frac{c_2}{Re}\|D\mathbf{w}_\alpha\|_0^2 + \frac{C}{Re}|Db(h)|_{L^\infty}\|D\mathbf{w}_\alpha\|_0\|\mathbf{w}_\alpha\|_0 \\ &\leq -\frac{c_2}{Re}\|D\mathbf{w}_\alpha\|_0^2 + \frac{\varepsilon c_2}{Re}\|D\mathbf{w}_\alpha\|_0^2 + \frac{C}{\varepsilon c_2 Re}\|Db(h)\|_{r_1}^2\|\mathbf{w}_\alpha\|_0^2 \end{aligned} \tag{B.31}$$

Via Sobolev inequalities and Cauchy's inequality with ε, and since $\nabla\cdot\mathbf{w} = 0$, the third term becomes

$$\begin{aligned} -\lambda(a(h)\nabla\rho_\alpha, \mathbf{w}_\alpha) &\leq C\lambda\|\bar{\rho}_\alpha\|_0|Da(h)|_{L^\infty}\|\mathbf{w}_\alpha\|_0 \\ &\leq \varepsilon\|\bar{\rho}_\alpha\|_0^2 + \frac{C\lambda^2}{\varepsilon}\|Da(h)\|_{r_1}^2\|\mathbf{w}_\alpha\|_0^2 \end{aligned} \tag{B.32}$$

Since $\nabla\cdot\mathbf{w} = 0$, the fourth term in (B.20) becomes

$$(c\nabla\Delta\rho_\alpha, \mathbf{w}_\alpha) = -(c\Delta\rho_\alpha, \nabla\cdot\mathbf{w}_\alpha) = 0 \tag{B.33}$$

The final term in inequality (B.29) becomes

$$\begin{aligned} (\mathbf{H}_\alpha, \mathbf{w}_\alpha) = \frac{1}{Re}((b(h)\Delta\mathbf{w})_\alpha - b(h)\Delta\mathbf{w}_\alpha, \mathbf{w}_\alpha) - \lambda((a(h)\nabla\rho)_\alpha - a(h)\nabla\rho_\alpha, \mathbf{w}_\alpha) \\ -((\mathbf{v}\cdot\nabla\mathbf{w})_\alpha - \mathbf{v}\cdot\nabla\mathbf{w}_\alpha, \mathbf{w}_\alpha) + (\mathbf{F}_\alpha, \mathbf{w}_\alpha) \end{aligned}$$

where $|\beta| = 1$. This equation is bounded by

$$(\mathbf{H}_\alpha, \mathbf{w}_\alpha) \le \frac{1}{Re}\|(b(h)\Delta\mathbf{w})_\alpha - b(h)\Delta\mathbf{w}_\alpha\|_0\|\mathbf{w}_\alpha\|_0 + \lambda\|(a(h)\nabla\rho)_\alpha - a(h)\nabla\rho_\alpha\|_0\|\mathbf{w}_\alpha\|_0$$
$$+ \|(\mathbf{v}\cdot\nabla\mathbf{w})_\alpha - \mathbf{v}\cdot\nabla\mathbf{w}_\alpha\|_0\|\mathbf{w}_\alpha\|_0 + \|\mathbf{F}_\alpha\|_0\|\mathbf{w}_\alpha\|_0 \tag{B.34}$$

Now applying the low-norm commutator estimate from Lemma B.1 gives

$$\|(b(h)\Delta\mathbf{w})_\alpha - b(h)\Delta\mathbf{w}_\alpha\|_0 \le C\|Db(h)\|_{k_1}\|\Delta\mathbf{w}\|_{k-1}$$
$$\|(a(h)\nabla\rho)_\alpha - a(h)\nabla\rho_\alpha\|_0 \le C\|Da(h)\|_{k_1}\|\nabla\rho\|_{k-1}$$
$$\|(\mathbf{v}\cdot\nabla\mathbf{w})_\alpha - \mathbf{v}\cdot\nabla\mathbf{w}_\alpha\|_0 \le C\|D\mathbf{v}\|_{k_1}\|\nabla\mathbf{w}\|_{k-1}$$

where $k = |\alpha|$ and $k_1 = max\{k-1, 2\}$. Now applying the low-norm commutator estimates and Cauchy's inequality with ε to inequality (B.34) we obtain

$$\begin{aligned}(\mathbf{H}_\alpha, \mathbf{w}_\alpha) &\le \frac{C}{Re}\|Db(h)\|_{k_1}\|\Delta\mathbf{w}\|_{k-1}\|\mathbf{w}_\alpha\|_0 + C\lambda\|Da(h)\|_{k_1}\|\nabla\rho\|_{k-1}\|\mathbf{w}_\alpha\|_0 \\ &\quad + C\|D\mathbf{v}\|_{k_1}\|\nabla\mathbf{w}\|_{k-1}\|\mathbf{w}_\alpha\|_0 + \|\mathbf{F}_\alpha\|_0\|\mathbf{w}_\alpha\|_0 \\ &\le \frac{\varepsilon c_2}{Re}\|\Delta\mathbf{w}\|_{k-1}^2 + \varepsilon\|\nabla\rho\|_{k-1}^2 + C\|D\mathbf{v}\|_{k_1}\|\nabla\mathbf{w}\|_{k-1}\|\mathbf{w}_\alpha\|_0 + \frac{1}{2}\|\mathbf{F}_\alpha\|_0^2 \\ &\quad + \frac{C}{\varepsilon}(\frac{1}{c_2 Re}\|Db(h)\|_{k_1}^2 + \lambda^2\|Da(h)\|_{k_1}^2)\|\mathbf{w}_\alpha\|_0^2 + \frac{1}{2}\|\mathbf{w}_\alpha\|_0^2\end{aligned} \tag{B.35}$$

Combining inequalities (B.30), (B.31), (B.32), (B.33), and (B.35) with (B.29) gives

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}_\alpha\|_0^2 + \frac{2c_2}{Re}\|\nabla\mathbf{w}_\alpha\|_0^2 &\le \frac{2\varepsilon c_2}{Re}\|D\mathbf{w}_\alpha\|_0^2 + 2\varepsilon\|\bar{\rho}_\alpha\|_0^2 + \frac{2\varepsilon c_2}{Re}\|\Delta\mathbf{w}\|_{k-1}^2 + 2\varepsilon\|\nabla\rho\|_{k-1}^2 \\ &\quad + C\|D\mathbf{v}\|_{k_1}\|\nabla\mathbf{w}\|_{k-1}\|\mathbf{w}_\alpha\|_0 + \|\mathbf{F}_\alpha\|_0^2 \\ &\quad + C(1 + |f|_{L^\infty} + \frac{1}{\varepsilon}(\frac{1}{c_2 Re}\|Db(h)\|_{k_1}^2 + \lambda^2\|Da(h)\|_{k_1}^2))\|\mathbf{w}_\alpha\|_0^2\end{aligned} \tag{B.36}$$

Summing inequality (B.27) and (B.36) over $0 \le |\alpha| \le r$ and applying Poincaré's inequality gives

$$\begin{aligned}&\frac{d}{dt}\|\mathbf{w}\|_r^2 + \frac{2c_2(1-\varepsilon-C\varepsilon)}{Re}\|\nabla\mathbf{w}\|_r^2 \\ &\quad \le C\varepsilon\|\nabla\rho\|_{r-1}^2 + C(\varepsilon+1)\|\mathbf{F}\|_r^2 \\ &\quad\quad + C(1 + |f|_{L^\infty} + \|D\mathbf{v}\|_{r_1} + \frac{1}{\varepsilon}(1 + \frac{1}{c_2 Re}\|Db(h)\|_{r_1}^2 + \lambda^2\|Da(h)\|_{r_1}^2))\|\mathbf{w}\|_r^2\end{aligned}$$

Substituting the estimate for $\|\nabla\rho\|_{r-1}^2$ from Lemma B.2 yields

$$\begin{aligned}
&\frac{d}{dt}\|\mathbf{w}\|_r^2+\frac{2c_2(1-\varepsilon-C\varepsilon)}{Re}\|\nabla\mathbf{w}\|_r^2\\
&\quad\le\frac{C\varepsilon}{\lambda^2c_1^2}[\frac{1}{Re^2}\|Db(h)\|_{r_1}^2\|\Delta\mathbf{w}\|_{r-2}^2+\|\mathbf{v}\cdot\nabla\mathbf{w}\|_{r-1}^2+\|\mathbf{F}\|_{r-1}^2]\\
&\quad\quad+C(\varepsilon+1)\|\mathbf{F}\|_r^2+C[1+|f|_{L^\infty}+\|D\mathbf{v}\|_{r_1}\\
&\quad\quad+\frac{1}{\varepsilon}(1+\frac{1}{c_2Re}\|Db(h)\|_{r_1}^2+\lambda^2\|Da(h)\|_{r_1}^2)]\|\mathbf{w}\|_r^2\\
&\quad\le\frac{C\varepsilon}{\lambda^2c_1^2Re^2}\|Db(h)\|_{r_1}^2\|\mathbf{w}\|_r^2+\frac{C\varepsilon}{\lambda^2c_1^2}\|\mathbf{v}\|_{r-1}^2\|\mathbf{w}\|_r^2\\
&\quad\quad+\frac{C\varepsilon}{\lambda^2c_1^2}\|\mathbf{F}\|_{r-1}^2+C(\varepsilon+1)\|\mathbf{F}\|_r^2\\
&\quad\quad+C[1+|f|_{L^\infty}+\|D\mathbf{v}\|_{r_1}+\frac{1}{\varepsilon}(1+\frac{1}{c_2Re}\|Db(h)\|_{r_1}^2+\lambda^2\varepsilon^2c_1^2)]\|\mathbf{w}\|_r^2\\
&\quad\le C(\varepsilon+1)\|\mathbf{F}\|_r^2+C[1+|f|_{L^\infty}+\|D\mathbf{v}\|_{r_1}+\frac{\varepsilon}{\lambda^2c_1^2}\|\mathbf{v}\|_{r-1}^2\\
&\quad\quad\quad+\frac{1}{\varepsilon}(1+(\frac{\varepsilon^2}{\lambda^2c_1^2Re^2}+\frac{1}{c_2Re})\|Db(h)\|_{r_1}^2)+\lambda^2\varepsilon c_1^2]\|\mathbf{w}\|_r^2
\end{aligned}$$

where C depends on r.

Using Gronwall's inequality gives

$$\begin{aligned}
&\|\mathbf{w}\|_r^2+\frac{c_2}{Re}\int_0^T\|D\mathbf{w}\|_r^2d\tau\\
&\quad\le e^{\int_0^T C(1+|f|_{L^\infty}+\|D\mathbf{v}\|_{r_1}+\frac{\varepsilon}{\lambda^2c_1^2}\|\mathbf{v}\|_{r-1}^2+\frac{1}{\varepsilon}+(\frac{\varepsilon}{\lambda^2c_1^2Re^2}+\frac{1}{\varepsilon c_2Re})\|Db(h)\|_{r_1}^2+\lambda^2\varepsilon c_1^2)d\tau}[\|\mathbf{w}_0\|_r^2\\
&\quad\quad+\int_0^T C(\varepsilon+1)\|\mathbf{F}\|_r^2d\tau]
\end{aligned}$$

where ε is sufficiently small so that $\frac{2c_2(1-\varepsilon-C\varepsilon)}{Re}\ge\frac{c_2}{Re}$. We used the facts that $(\frac{1}{Re})\|Db(h)\|_{r_1,T}^2\le\varepsilon$ and $\|Da(h)\|_{r_1,T}\le\varepsilon c_1$. We remark that $\frac{\varepsilon}{\delta}=O(1)$. And T is sufficiently small so that
$CT(1+|f|_{L^\infty,T}+\|D\mathbf{v}\|_{r_1,T}+\frac{\varepsilon}{\lambda^2c_1^2}\|\mathbf{v}\|_{r-1,T}^2+\frac{1}{\varepsilon}+(\frac{\varepsilon}{\lambda^2c_1^2Re^2}+\frac{1}{\varepsilon c_2Re})\|Db(h)\|_{r_1,T}^2+\lambda^2\varepsilon c_1^2)=O(1)$.
We obtain

$$\|\mathbf{w}\|_r^2+\frac{c_2}{Re}\int_0^T\|D\mathbf{w}\|_r^2d\tau\le C[\|\mathbf{w}_0\|_r^2+\int_0^T\|\mathbf{F}\|_r^2d\tau]$$

where C depends on r which completes the proof.