CALCULATION OF LAPLACE AND HELMHOLTZ POTENTIALS IN TWO-PHASE PROBLEMS

A Thesis

by

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This thesis meets the standards for scope and quality of
Texas A&M University-Corpus Christi and is hereby approved.

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ABSTRACT

In this thesis, two phase models in a magnetostatics context using the Maxwell-Maxwell (MM) model and the Maxwell-London (ML) model are investigated. The vector equations are transformed in terms of scalar potentials leading to mixed boundary value problems for Laplace-Laplace and Laplace Helmholtz equations in the respective cases. Exact analytic solutions for the exterior and interior potentials $\Phi^e(r,\theta,\phi)$ and $\Phi^i(r,\theta,\phi)$, where r,θ,ϕ are the spherical coordinates, are obtained as infinite series and in closed forms for the MM model. The general solutions are found as a theorem. Several illustrative examples for specific externally imposed magnetic fields including a magnetic monopole and dipole are discussed based on our analytic solutions. It is shown that the magnetic permeability parameter $k = \frac{\mu^e}{\mu^e+\mu^i}$, where μ^e and μ^i are magnetic permeabilities in the exterior and interior phases, has a significant impact on the magnetic induction fields and the forces acting on the sphere. A new relation for the multipole coefficients of the external phase is derived as well. Exact solutions for the ML model involving a superconducting sphere are derived in terms of the magnetic flux density functions $\Psi^e(r,\theta)$ and $\Psi^i(r,\theta)$ in the respective phases. The general solutions are established as a theorem for this model as well. The non-dimensional penetration depth parameter λ is found to dictate the induction fields in ML model. Our results are of interest in various topics in mathematical physics where two phase models are used[1].

[1]Parts of this work were presented in the following conferences.
1. *American Physical Society - Division of Fluid Dynamics (APS-DFD)*, Nov. 20-22, Portland, Oregon
2. *Texas Differential Equations Conference*, April 09, 2017, Texas State University, San Marcos
3. *Coastal Bend Mathematics and Statistics Symposium (CBMSS)*, March 26, 2016, Texas A&M University, Corpus Christi

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DEDICATION

I dedicate my thesis to my children Rama Shokani and Fahad Shokani.

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CHAPTER I: INTRODUCTION

Problems involving two phase media occur in a variety of situations in science and technology. A dielectric sphere embedded in a different dielectric background [10, 21], a magnetic sphere in an external magnetic field [7, 12, 13], and a polymeric fiber suspended in an aqueous solution [8] are a few examples of two phase models. Better understanding of such models requires theoretical studies involving the calculation of electric, magnetic and velocity fields for the two phases. These calculations can be successfully done via finding suitable scalar functions commonly known as *potential functions* or *potentials*. Thus, the determination of potentials in a two phase media is a classic and important problem and has been of interest to mathematicians, physicists, and engineers. For instance, measurements of electrical conductivity [27] and the dielectric constant [18] in electrostatics, or the estimation of magnetic permeability in magnetostatics [18] are based on the knowledge of potentials in the two phases. Similar studies modeling the flow through porous media [8] and thermal conduction and diffusion processes [15] further highlight the significance of two-phase problems. Recent studies in the calculation of Clausius-Mossotti factors [9] and finding the ion structure near a core-shell dielectric nanoparticle [16] indicate a strong interest in the subject. Generally, two-phase models lead to mathematical problems involving second order partial differential equations with mixed type boundary conditions.

The two phase problems have roots in basic electrostatics [18, 10] and have been investigated based on various circumstances. They also occur in numerous other fields including magnetostatics, thermodynamics and fluid dynamics. The thermal conductivity, dielectric constant, magnetic permeability, and diffusion properties of two-phase objects can often be formulated in ways precisely analogous to those used in the treatment of the problems in electrostatics. There is therefore an overlap in the literature for all these phenomena, in which theory and experimental results from one can be readily applied to another. The analogy between hydrodynamics and superconductivity, for example, is explained and exploited by Palaniappan [22] and Trombley and Palaniappan [26].

The significance of the computation of potential functions leading to the calculation of forces and moments is clearly demonstrated in these studies. Our main focus of this thesis is to determine the potentials in a two-phase medium in the context of magnetostatics, although the results can be translated to other fields by analogy.

The mathematical treatment of two phase models can be challenging and may require sophisticated mathematical tools. As mentioned in the preceding paragraph, in this thesis we investigate boundary-value problems in two-phase media that are of interest in the context of magnetostatics. A typical model of a two-phase problem is illustrated in Figure 1.1. An isolated magnetic sphere of radius a and permeability μ^i is placed in an external magnetic field generated in a medium of permeability μ^e . The fields are assumed to be independent of time in the two phases. We consider the following two types of problems in the magnetostatic environment, namely,

- A magnetized sphere in an external arbitrary magnetic field which is the Maxwell-Maxwell (MM) model

- A superconducting sphere suspended in an external axisymmetric magnetic field which is the Maxwell-London (ML) model

The first problem requires the solution of the Laplace equations (harmonic potentials), and the second model demands the solution of the Laplace equation (harmonic potential) for the exterior phase and the solution of the Helmholtz equation (Helmholtz potential) for the interior phase. The governing partial differential equations in vector forms are based on Maxwell's and London's formulations [18, 14] that are typically used in magnetostatics. We transform the vector boundary value problems into a scalar problem of finding a potential function by the use of a suitable transformation. The scalar functions which are called potentials are key to study the respective two-phase problems. The magnetic induction exterior to the spherical boundary satisfies Maxwell's equation in both problems. But in the interior domain we assume Maxwell's equation (Laplace's equation) for the MM model: magnetized sphere problem while London's equation (Helmholtz's equation) for the superconducting sphere problem. For some special external fields exact solutions are available for the two models. For example, the analytic solution for a magnetized sphere placed in a uniform magnetic field is given in standard text books [18, 10]. Matute [17] analyzed the Maxwell-London model in the case of a superconducting sphere in a *c*onstant magnetic field. Here we attempt to explore general solutions for Maxwell-Maxwell and Maxwell-London models for a sphere placed in an arbitrary magnetic field. We derive closed form results for potentials in the two phases satisfying the relevant boundary conditions and use them to determine the forces acting on the sphere in some situations.

Spherical objects are the simplest of all the quadric surfaces in three dimensions possessing various symmetries. It is known that the Laplace equation is separable in the spherical coordinate system, which makes several problems tractable. The use of spherical harmonics [3, 31, 5, 4, 6] for understanding electromagnetic systems has a long and fruitful history. The spherical coordinate system allows one to characterize a system in terms of a multiple or spherical harmonic expansion. This approach is well understood, and has dominated the literature for years. Moreover, for purely spherical geometries, this is the most appropriate method. Furthermore, the standard boundary conditions such as Dirichlet, Neumann and mixed type can be handled in a convenient way with the spherical coordinate system. For these reasons we have chosen the boundary of one phase to be a sphere and then provide a fairly detailed analysis.

Maxwell's and London's equations are in vector form for the magnetic fields that are relatively hard to solve. For the MM model, the magnetic field is both solenoidal (divergence-free) and irrotational (curl-free) and therefore a potential function formulation is directly possible as demonstrated in the literature (for instance, see [18, 10]). But in the ML model, the magnetic field is solenoidal but not irrotational. As shown in the fluid dynamics context [8], if the field is axi-symmetric with respect to the z-axis then it is possible to represent the magnetic field as the curl of a potential function. This potential function is known as the *S*tokes stream function in fluid dynamics. Surprisingly, it does not seem to be widely recognized in magnetostatic and electromagnetic problems that can be characterized by similar scalar functions. Various studies of axisymmetric problems in magnetostatics have used other relatively complex methods [34, 28, 23] to derive results even in simple cases. Here we show that the equations for magnetic problems involving superconducting spheres using the ML model can be formulated and solved in terms of a magnetic flux function analogous to the stream-function. The flux function shows magnetic lines of force in the field just as the stream function portrays the streamlines in the fluid dynamical case.

The thesis is organized as follows. In chapter II, we provide a short description of the two phase models and their mathematical formulations. The vector equations for the magnetic fields for the Maxwell-Maxwell model are given in section 2.1. The potential functions setting along with the mixed boundary conditions on the spherical boundary for MM model are discussed. Some special cases of internal and external conditions of our MM model are also listed. The vector equations for the Maxwell-London model are given in section 2.2. The boundary value problem for the ML model for a superconducting sphere in an external field is stated in terms of the magnetic flux density function. The solution of the Laplace equation in spherical coordinates is described in section 2.3. The solution of the axisymmetric Helmholtz equation in spherical coordinates is given in section 2.4.

Chapter III contains a number of analytic results for the MM model. The calculation of the scalar potentials in the exterior and interior phases for the this model is explained in section 3.1. The general solutions in the two phases are given in infinite series form first. Then the sum of the series is found and the general results are expressed in closed forms containing integrals. Our general results are stated and proved in the form of a theorem for the MM model. Exact solutions for constant and linear magnetic inductions in the presence of a magnetized sphere are discussed in sections 3.2 and 3.3. The potential plots for these fields and their variations are shown in these sections. Analytic solutions for the magnetic pole and the corresponding image system are derived in section 3.4. The calculation of the force acting on a sphere due to a magnetic pole is given in

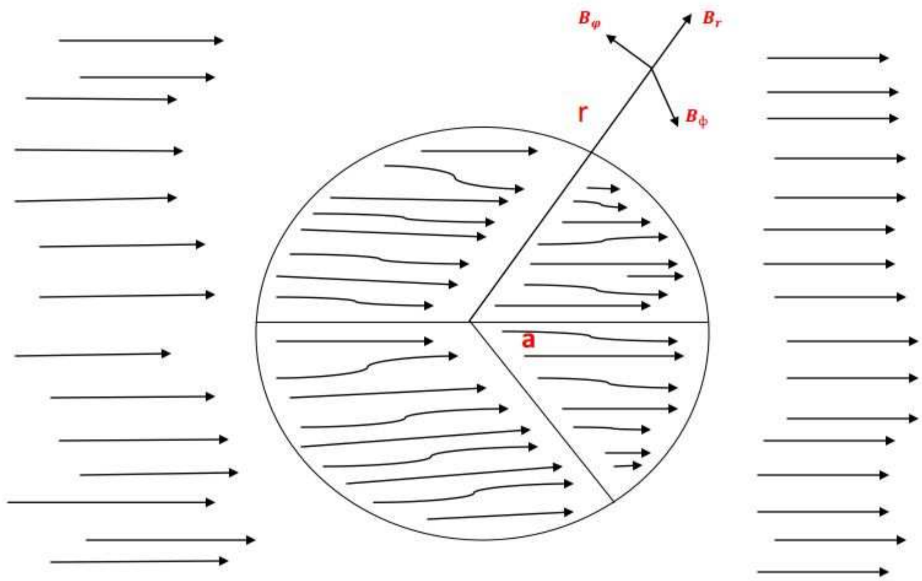

Figure 1.1
Typical magnetic field lines in a two phase medium

section 3.5 together with graphical illustrations. The corresponding results for a magnetic dipole oriented along radial and transverse directions are given in the section 3.6. An interesting *n*ew relation for the multipole coefficients in the exterior phase of the MM model is derived in the section 3.7.

Analytic solutions for the Maxwell-London model are derived in infinite series forms in Chapter IV. The solutions in the exterior and interior flux density functions contain Bessel functions of fractional order. A theorem for the general solutions for an arbitrary axisymmetric external magnetic field in the presence of a superconductor is stated and proved. The exact solutions for constant and linear magnetic induction field in the presence of a superconducting sphere are described in the sections 4.1 and 4.2. The field plots for a superconducting sphere in a constant and linear fields are also included in those sections. Finally, our main findings are summarized in Chapter V.

Some Definitions:

Inverse point: Let $P(x,y,z)$ be a point outside the sphere of radius a and let r be the distance between the center of the sphere at $(0,0,0)$ and P. Then $r' = \frac{a^2}{r}$ is defined to be the distance from

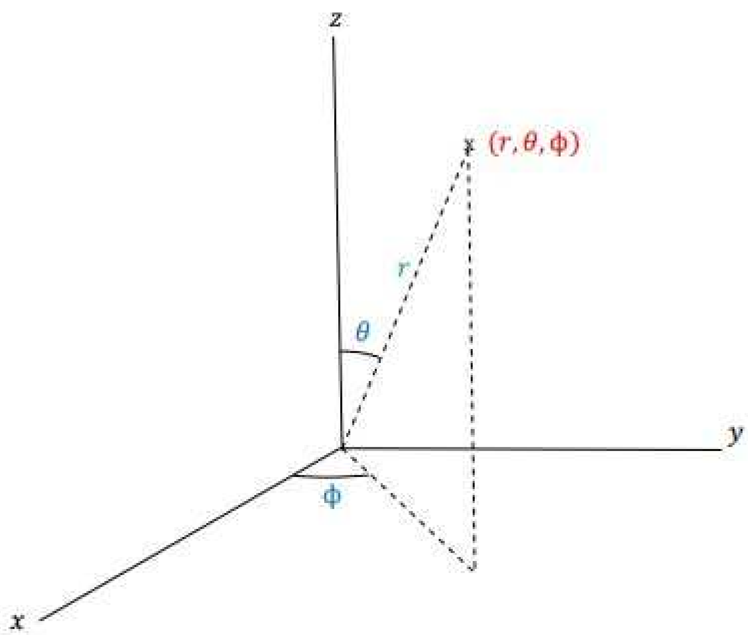

Figure 1.2
The spherical polar coordinate system

the center to the image point inside the sphere such that $rr' = a^2$ and is known as the *inverse point*.

Image system: Let $\Phi_0(x,y,z)$ be a given potential function for the solution of the Laplace equation in the absence of boundaries. Due to the linearity of Laplace equation, if a boundary is introduced, then the potential of the modified solution satisfying the given boundary conditions can be written as

$$\Phi = \Phi_0 + \Phi_1$$

Here Φ_1 is said to be the image system.

Spherical polar coordinate system: We use the spherical coordinates as shown in Figure 2.3. According to this notation the conversion from cartesian coordinates (x,y,z) to the spherical coordinates (r,θ,ϕ) is

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Legendre polynomials: The associated Legendre polynomials are given by [7, 10]

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m}[P_l(x)]$$

$$P_l(x) = \frac{m}{2^l l!} \frac{d^l}{dx^l}[(x^2-1)^l]$$

$$P_0^0(x) = 1$$

$$P_1^0(x) = x$$

$$P_1^1(x) = -1(1-x^2)^{\frac{1}{2}}$$

$$P_2^0(x) = \frac{1}{2}(3x^2-1)$$

$$P_2^1(x) = -3x(1-x^2)\frac{1}{2}$$

$$P_2^2(x) = 3(1-x^2)$$

Modified spherical Bessel function of the first kind: The modified spherical Bessel functions of the first kind with argument z are defined as [1, 7]

$$f_n(z) \equiv f_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z)$$

where $I_n(z)$ is the modified Bessel function of the first kind [1]. Some special values are

$$f_0(z) = \frac{\sinh(z)}{z}$$

$$f_1(z) = \frac{z\cosh(z) - \sinh(z)}{z^2}$$

$$f_2(z) = \frac{(z^2+3)\sinh(z) - 3z\cosh(z)}{z^2}$$

and so on.

Axisymmetric field: An *axisymmetric field* is the one in which the field is independent of the azimuthal angle ϕ in spherical coordinate (r,θ,ϕ). In this case the components of the field B_r and B_θ do not depend on the ϕ coordinate and $B_\phi = 0$. In our investigation we take z-axis to be the axis of symmetry.

Various approaches for studying magnetic, electrical, thermal and elastic properties of two-phase media have been introduced since Maxwell's seminal work on spherical particle suspensions more than a century ago [18]. In particular, the twin-phase models in basic electro- and magneto-statics are essentially based on Maxwell's vector differential equations for the electric and magnetic fields, respectively. In the Maxwell-Maxwell two-phase model, the magnetic fields in the exterior and interior phases are assumed to be continuous across the boundary in question. Although the MM model has been used to find solutions of various external fields, a systematic approach is lacking. If the magnetic fields are allowed to cross the boundary surfaces, then the London theory [14] is found to be better than Maxwell equations in the interior phase. The boundary surface in the latter case is the so called *s*uperconducting surface and the corresponding mathematical problem for ML model has been discussed only for a special case, namely a constant field (see for instance, [17, 20, 24]). In this thesis, we will deal with both MM and ML models with a view to provide a unique approach leading to analytical solutions for arbitrary magnetic fields imposed externally. We remark that the MM model contains a variety of special cases including the perfectly superconducting geometry discussed recently by Trombley and Palaniappan [26]. Below we document the mathematical formulation for the MM and ML two-phase models involving spherical boundaries.

2.1 Magnetized sphere in an external magnetic field: the MM model

Let us consider a magnetic/magnetized sphere of permeability μ^i placed in a magnetic field acting in a medium of permeability μ^e (see Figure 2.2). The magnetic induction inside the sphere of radius a is denoted by $\mathbf{B}^i$ while the induction outside is represented by $\mathbf{B}^e$, respectively. The vector field equations in the two phases, in simplified forms, are given by [18]

$$outside: \qquad \nabla^2\mathbf{B}^e = 0, \qquad \nabla\cdot\ \mathbf{B}^e = 0 \tag{2.1}$$

$$inside: \qquad \nabla^2\mathbf{B}^i = 0, \qquad \nabla\cdot\mathbf{B}^i = 0 \tag{2.2}$$

where the Laplace operator in cartesian coordinates is $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}$ in three dimensions. By making the change from cartesian (x,y,z) to spherical coordinates (r,θ,ϕ) and converting all the derivatives one can obtain the Laplace operator in spherical coordinates. The first part of these equations follows from the Maxwell theory while the second equations are the incompressibility condition (solenoidal property) of the magnetic induction fields. The determination of the exterior and interior fields $\mathbf{B}^e$ and $\mathbf{B}^i$ subject to boundary conditions requires solving the vector harmonic equations given in part 1 of $(2.1)-(2.2)$. The resulting analysis is complicated and so one seeks a scalar formulation of the problem. We note that the magnetic induction is also irrotational [18], that is,

$$\nabla\times\mathbf{B}^e = \nabla\times\mathbf{B}^i = 0$$

This implies that there exists scalar functions Φ^e and Φ^i called the *potential functions* such that

$$\mathbf{B}^e = -\nabla\Phi^e \tag{2.3}$$

$$\mathbf{B}^i = -\nabla\Phi^i \tag{2.4}$$

in the two phases, respectively. The negative sign in front of the gradient operator is due to the physical reason that the potential drops from a higher level to a lower level. Now the application of incompressibility condition of the magnetic induction (part 2 of (2.1) and (2.2)) to (2.3) and (2.4) yields

$$\nabla^2\Phi^e = 0 \tag{2.5}$$

$$\nabla^2\Phi^i = 0 \tag{2.6}$$

in the two phases. Therefore, the vector equations (2.1) and (2.2) for the MM model reduce to solving the scalar Laplace equations (2.5) and (2.6) for the potential functions subject to the boundary conditions. For a magnetized sphere embedded in another medium the appropriate boundary conditions are according to [10]

$$\Phi^e = \Phi^i \qquad \text{on} \qquad r = a \tag{2.7}$$

$$\mu^e \frac{\partial \Phi^e}{\partial r} = \mu^i \frac{\partial \Phi^i}{\partial r} \qquad \text{on} \qquad r = a \tag{2.8}$$

Note that (2.7) is a Dirichlet type condition and (2.8) represents a Neumann type condition on the surface of the spherical boundary [19]. Thus, the two phase mathematical *Maxwell-Maxwell model* (MM model) for a magnetized sphere placed in an external magnetic field reduces to solving the *mixed boundary-value problem* (BVP) for the Laplace equations given by (2.5) - (2.8). The solutions of the mathematical boundary value problem will yield the corresponding potentials for the two phases. The magnetic induction in the respective phases are then determined using (2.3) and (2.4), after a straightforward differentiation. We return to the calculations of the potentials Φ^e and Φ^i via analytic solutions of the BVP along with a discussion of a number of examples in chapter 3 . We will use the following non-dimensional permeability parameter defined by

$$k = \frac{\mu^e}{\mu^e + \mu^i} \tag{2.9}$$

in our analysis.

Although our discussion of the two-phase problem stated in this subsection is in the context of magnetostatics we remark that it can be readily translated to other related topics in mathematical physics including

- A sphere of magnetic permeability μ in a vacuum [7]:

  $\mu^e = 1, \qquad \mu^i = \mu, \qquad k = \frac{1}{\mu+1}$

- Dielectric sphere in another dielectric medium [21]:

  $\mu^e = \varepsilon_1, \qquad \mu^i = \varepsilon_2$

- Hydrodynamic and perfectly superconducting case [29, 26]: $k = 1$

- Heat conduction problem for a sphere [32, 33]: $\mu^e = k_1, \mu^i = k_2$, where k_1 and k_2 are the thermal conductivities in the respective phases

- Darcy flow past a porous sphere [8]

- $\mu^i < 0$ represents unphysical situation

2.2 Superconducting sphere in an external magnetic field: the ML model

As our second model, we consider a superconducting sphere of radius a placed in an external magnetic field. This problem corresponds to Maxwell-London model (ML model). The governing vector equations in this case are the Maxwell-London equations [14] given by

$$outside: \qquad \nabla^2 \mathbf{B}^e = 0, \qquad \nabla \cdot \mathbf{B}^e = 0 \tag{2.10}$$

$$inside: \qquad \nabla^2 \mathbf{B}^i = \frac{1}{\lambda^2}\mathbf{B}^i \qquad \nabla \cdot \mathbf{B}^i = 0 \tag{2.11}$$

where λ is the phenomenological London parameter measuring the penetration depth of the magnetic field in the superconductor. It should be pointed out that in the limit $\lambda \to \infty$ the London equations reduce to the Maxwell equations. According to the London theory [14], the exterior magnetic field is allowed to enter inside and vice-versa. Note that the magnetic inductions satisfy the vector Laplace and Helmholtz equations in the exterior and interior phases, respectively. In contrast to the MM model, the magnetic induction in the interior phase is *not* irrotational and so a relation as in (2.4) is not possible for the London equations. However, the incompressibility conditions in the two phases imply that we can define

$$\mathbf{B}^e = \nabla \times \left(\Psi^e \hat{\mathbf{e}}_\phi \right) \tag{2.12}$$

$$\mathbf{B}^i = \nabla \times \left(\Psi^i \, \hat{\mathbf{e}}_\phi \right) \tag{2.13}$$

where $\Psi^e(r,\theta)$ and $\Psi^i(r,\theta)$ are known as the axisymmetric magnetic flux density functions and $\hat{\mathbf{e}}_\phi$ is the unit vector in ϕ direction. In hydrodynamical context the function Ψ is known as the Stokes stream function [8]. In the axisymmetric case the fields in the planes parallel to the z-axis (the axis of symmetry) are the same (see [8] for instance). Therefore, the Ψ^e and Ψ^i are independent of the angle ϕ. By the use of these scalar functions defined in (2.12) and (2.13) in (2.10) and (2.11) we obtain the following partial differential equations in the two phases.

$$D^2\Psi^e = 0 \tag{2.14}$$

$$D^2\Psi^i - \frac{1}{\lambda^2}\Psi^i = 0 \tag{2.15}$$

where the axisymmetric Laplace operator D^2 in spherical coordinates is given by [8]

$$D^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Here again, the *Maxwell-London model* vector equations (2.10) and (2.11) reduce to solving the scalar axisymmetric Laplace and Helmholtz equations for the exterior and interior potentials. Now for a superconducting sphere suspended in a given external magnetic field, the appropriate boundary conditions are the continuity of the induction components across the surface $r = a$ of the sphere [17]. This in turn leads to the boundary conditions in terms of the axisymmetric scalar potentials

$$\Psi^e = \Psi^i \qquad on \qquad r = a \tag{2.16}$$

$$\frac{\partial\Psi^e}{\partial r} = \frac{\partial\Psi^i}{\partial r} \qquad on \qquad r = a \tag{2.17}$$

As in the MM model, the above two conditions represent Dirichlet and Neumann type conditions at the surface of a superconductor for the ML model. Therefore, the two-phase problem, for a superconducting sphere embedded in an external magnetic field reduces to solving the mixed boundary value problem for the axisymmetric Laplace and Helmholtz equations, given in (2.14) - (2.17).

The solution of the mathematical boundary value problem will yield the corresponding potential functions for the two phases. The magnetic induction in the exterior and interior phases are then

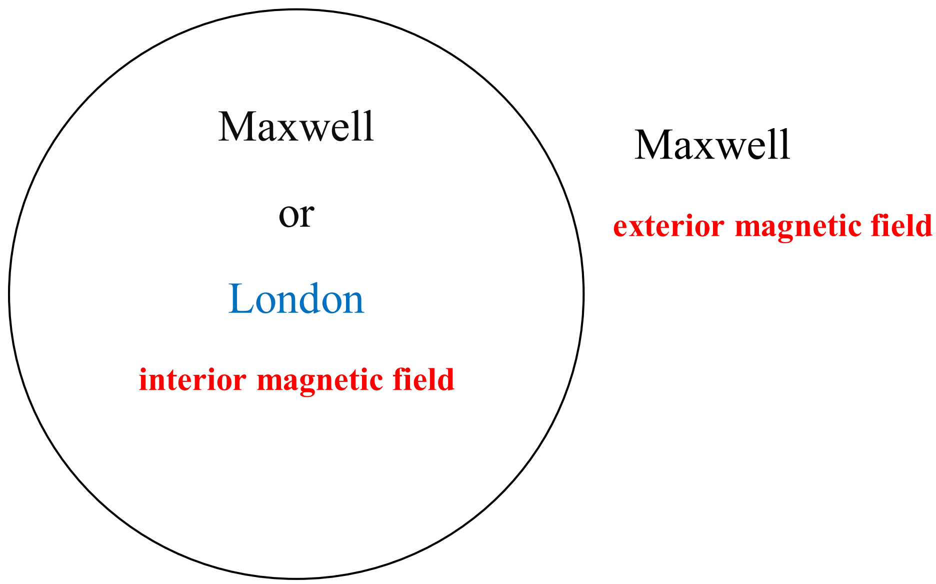

Figure 2.3
Representation of MM and ML models

determined from (2.12) and (2.13). We will return to the calculation of exact solutions for the potentials Ψ^e and Ψ^i and investigate various examples in chapter IV.

In the following sections we provide general solutions for the Laplace equation (2.5) or (2.6) and the axisymmetric Helmholtz equation (2.15) in spherical coordinates, The derivation of these solutions can be found elsewhere [8, 7], but here we record the final solutions along with the key steps only.

2.3 Solution of the Laplace equation in spherical coordinates

The Laplace equation for a function Φ written in spherical coordinates is

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{\csc^2\theta}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}=0 \tag{2.18}$$

Using the separation of variables method for solving partial differential equations [19], we assume

$$\Phi(r,\theta,\phi)=F(r)G(\theta)H(\phi) \tag{2.19}$$

Substituting (2.19) into (2.18) yields

$$F''GH+\frac{2}{r}F'GH+\frac{1}{r^2}FG''H+\frac{1}{r^2}\cot\theta FG'H+\frac{1}{r^2}\csc^2\theta FGH''=0 \tag{2.20}$$

Thus we have

$$\frac{F''}{F}+\frac{2}{r}\frac{F'}{F}=-\frac{1}{r^2}\frac{G''}{G}-\frac{1}{r^2}\cot\theta\frac{G'}{G}=-\frac{1}{r^2}\csc^2\theta\frac{H''}{H}=\eta^2 \tag{2.21}$$

where η^2 is a separation constant. The separation technique leads to the following equations for F, G and H:

$$r^2F''+2rF'-\eta^2F=0 \tag{2.22}$$

$$\sin^2\theta G''+\cos\theta\sin\theta G'+\eta^2G=0 \tag{2.23}$$

$$H''+\eta^2H=0 \tag{2.24}$$

The last equation gives, according to the basic solutions of the form [19]

$$H(\phi)=A_{nm}\cos m\phi+B_{nm}\sin m\phi \tag{2.25}$$

It can be shown that [19]

$$\eta^2=n(n+1)=m^2 \tag{2.26}$$

and so suitable solutions of (2.22) and (2.23) are of the form

$$F(r)=r^n \qquad \text{or} \qquad F(r)=r^{-(n+1)} \tag{2.27}$$

$$G(\theta)=P_n^m(\cos\theta) \tag{2.28}$$

where $P_n^m(\cos\theta)$ is a Legendre polynomial of the second kind [7, 19]. Combining (2.25) - (2.28) we get

$$\Phi(r,\theta,\phi)=\sum_{n=0}^{\infty}[A_nr^n+B_nr^{-(n+1)}]P_n^m(\cos\theta)[A_{nm}\cos m\phi+B_{nm}\sin m\phi] \tag{2.29}$$

We remark that the constants A_{nm} and B_{nm} may also be absorbed in A_n and B_n.

2.4 Solution of the axisymmetric Helmholtz equation

The Helmholtz equation for a function Ψ with axial symmetry is

$$(D^2-\frac{1}{\lambda^2})\Psi=0 \tag{2.30}$$

Using the spherical polar coordinates form of the operator D^2, the above equation becomes

$$\frac{\partial^2\Psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{1}{\lambda^2}\Psi = 0 \tag{2.31}$$

where $\Psi = \Psi(r,\theta)$. By the use of separation of variables method for solving partial differential equations [19] we take the solution of the form

$$\Psi(r,\theta) = R(r)\Theta(\theta) \tag{2.32}$$

where $R(r)$ is a function of r and $\Theta(\theta)$ is a function of θ only. Substituting (2.32) into (2.31) and after multiplying by $\frac{r^2}{R(r)\Theta(\theta)}$ we get

$$\frac{r^2R''}{R} - \frac{1}{\lambda^2}r^2 = -\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} = \eta^2 \tag{2.33}$$

here η^2 is a separation constant. As shown in [19] the separation constant is given by $\eta^2 = n(n+1)$. Thus the equation for R and Θ become

$$r^2R'' - \left[\frac{r^2}{\lambda^2} + n(n+1)\right]R = 0 \tag{2.34}$$

$$\Theta'' - \cot\theta\Theta' + n(n+1)\Theta = 0 \tag{2.35}$$

We first solve (2.35) by the substitution method. Taking $\zeta = \cos\theta$ in (2.35) one obtains

$$\frac{d}{d\zeta}(1-\zeta^2)\frac{d\Theta(\zeta)}{d\zeta} + n(n+1)\Theta(\zeta) = 0 \tag{2.36}$$

The preceding equation is the well-known associated Legendre differential equation [7, 19]. The solution of this equation is the associated Legendre function $P_n^1(\zeta)$ of order n and index $m = 1$. Thus the solution of (2.35) can be written as

$$\Theta(\zeta) = P_n^1(\zeta) \qquad \text{that is} \qquad \Theta(\cos\theta) = P_n^1(\cos\theta) \tag{2.37}$$

Next, we note that (2.34) is the modified Bessel equation (after making a suitable change of variable). Its solution is written as [19]

$$R(r) = r\left[C_nI_{n+\frac{1}{2}}(\lambda r) + D_nK_{n+\frac{1}{2}}(\lambda r)\right] \tag{2.38}$$

where $I_{n+\frac{1}{2}}$ and $K_{n+\frac{1}{2}}$ are modified Bessel functions of fractional order. Combining (2.37) and (2.38) we obtain

$$\Psi(r,\theta) = r\sum_{n=0}^{\infty}\left[C_n I_{n+\frac{1}{2}}(\lambda r) + D_n K_{n+\frac{1}{2}}(\lambda r)\right] P_n^1(\cos\theta) \tag{2.39}$$

Equation (2.39) is the general solution of the axisymmetric Helmholtz equation [8]. In a similar way the solution of the axisymmetric Laplace equation can be derived, see for instance the exposition in [8]. In the ML model problem (see Chapter IV) we only need to use $I_{n+\frac{1}{2}}$ since it is finite at the origin and not $K_{n+\frac{1}{2}}$ which is finite at infinity [1].

CHAPTER III: Analytic Solutions for the Maxwell-Maxwell model (MM model)

3.1 Potentials for the Maxwell-Maxwell model

In this chapter we calculate the effect of placing a magnetic sphere of permeability μ^i into a magnetic field acting in a medium of permeability μ^e. This problem is the so called Maxwell-Maxwell model and the corresponding mathematical boundary value problem is stated in section 2.1. As discussed there, the underlying mathematical problem reduces to solving the Laplace equation in the exterior and interior region subject to the mixed boundary condition at the spherical boundary $r = a$. (see $(2.5) - (2.8)$ in 2.1). Below we derive exact analytic solutions for the potential functions Φ^e and Φ^i in spherical coordinates that yield magnetic induction fields in the two regions. The corresponding gradients will describe the resulting magnetic fields in the presence of a magnetized sphere.

There are numerous methods for solving mixed boundary value problem for the Laplace equation. The standard method of using spherical harmonics has been used for spherical boundaries. This technique involves an infinite series expansion in terms of harmonic functions and determine the coefficients based on the boundary conditions. The method of images [29], the Mellin transform method [30] and numerical techniques are some of the other approaches utilized in the literature. These techniques require a lot of guess work and approximation. Therefore, we use the spherical harmonic function method along with the convergence results for infinite series to determine Φ^e and Φ^i. We show below various steps in the construction of the solution to our problem in the two phases using spherical harmonic expansion method.

Let $\Phi_0(r,\theta,\phi)$ be a given potential in the absence of any boundaries where (r,θ,ϕ) are spherical coordinates. Since $\Phi_0(r,\theta,\phi)$ must satisfy the Laplace equation, it is a harmonic function. As explained in [10], a harmonic function in spherical coordinates can be expanded in an infinite series form as

$$\Phi_0(r,\theta,\phi) = \sum_{n=0}^{\infty} A_n r^n S_n(\theta,\phi) \tag{3.40}$$

where A_n is a constant coefficient and

$$S_n(\theta,\phi) = \sum_{m=0}^{n} P_n^m(\cos\theta)[\cos m\phi + \sin m\phi] \tag{3.41}$$

The term $r^n S_n(\theta,\phi)$ is called a spherical harmonic function of degree n. Below we find the exact solutions of the BVP (2.5) - (2.8) for the MM model.

Theorem: Let $\Phi_0(r,\theta,\phi)$ be an arbitrary potential field in the absence of any boundary. If a magnetized sphere of radius $r=a$ centered at the origin $(0,0,0)$ is introduced in the field of Φ_0 then the modified potentials for the MM model for the exterior and interior phases satisfying the mixed BVP (2.5) - (2.8) are given by

$$\Phi^e(r,\theta,\phi) = \Phi_0(r,\theta,\phi) + \sum_{n=0}^{\infty}\left[-(1-2k) + \frac{k(1-2k)}{n+k}\right]\frac{a^{2n+1}}{r^{n+1}} A_n S_n(\theta,\phi) \tag{3.42}$$

$$\Phi^i(r,\theta,\phi) = \sum_{n=0}^{\infty}\left[2k + \frac{k(1-2k)}{n+k}\right] r^n A_n S_n(\theta,\phi) \tag{3.43}$$

in series form. The closed form solutions are given by

$$\begin{aligned}\Phi^e(r,\theta,\phi) &= \Phi_0(r,\theta,\phi) - (1-2k)\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right) \\ &+ k(1-2k)r^{k-1}a^{-2k+1}\int_0^{\frac{a^2}{r}} R^{k-1}\Phi_0(R,\theta,\phi)\,dR. \end{aligned} \tag{3.44}$$

$$\Phi^i(r,\theta,\phi) = 2k\Phi_0(r,\theta,\phi) + \frac{k(1-2k)}{r^k}\int_0^r R^{k-1}\Phi_0(R,\theta,\phi)\,dR. \tag{3.45}$$

Proof: Let $\Phi_0(r,\theta,\phi)$ be a given potential in the absence of any boundary. Since Φ_0 is a harmonic function, it can expressed in a series as given in (3.40) - (3.41). When the magnetized sphere $r=a$ is introduced into the field of Φ_0 then the exterior potential can be taken as (see solutions of the Laplace equation in section 2.3)

$$\Phi^e(r,\theta,\phi) = \sum_{n=0}^{\infty}\left[A_n r^n + \frac{B_n}{r^{n+1}}\right] S_n(\theta,\phi) \tag{3.46}$$

In the interior phase the potential function satisfies the Laplace equation and so a suitable choice for Φ^i is

$$\Phi^i(r,\theta,\phi) = \sum_{n=0}^{\infty} C_n r^n S_n(\theta,\phi) \tag{3.47}$$

The constant coefficients B_n and C_n in (3.46)-(3.47) will be determined using the boundary conditions (2.7)-(2.8) for the Maxwell-Maxwell model given in section 2.1. Using (2.7) and (2.8) for Φ^e and Φ^i, one obtains

$$B_n = \left[\frac{n\mu^e - n\mu^i}{(n+1)\mu^e + n\mu^i}\right] a^{2n+1} A_n \tag{3.48}$$

and

$$C_n = \left[\frac{(2n+1)\mu^e}{(n+1)\mu^e + n\mu^i}\right] A_n. \tag{3.49}$$

Now the term in the square bracket in (3.48) can be written as

$$\frac{n\mu^e - n\mu^i}{(n+1)\mu^e + n\mu^i} = \frac{n(\mu^e - \mu^i)}{n(\mu^e + \mu^i) + \mu^e}. \tag{3.50}$$

Define

$$\frac{\mu^e}{\mu^e + \mu^i} = k. \tag{3.51}$$

Then

$$\frac{\mu^e - \mu^i}{\mu^e + \mu^i} = 2k - 1. \tag{3.52}$$

Now the right side of of (3.50) becomes

$$\frac{n(\mu^e - \mu^i)}{n(\mu^e + \mu^i) + \mu^e} = \frac{n(2k-1)}{n+k} \tag{3.53}$$

$$= (2k-1) - \frac{k(2k-1)}{n+k} \tag{3.54}$$

So B_n in (3.48) takes the form

$$B_n = \left[-(1-2k) + \frac{k(1-2k)}{n+k}\right] a^{2n+1} A_n. \tag{3.55}$$

As above the term inside the square bracket in (3.49) is written as

$$\frac{(2n+1)\mu^e}{(n+1)\mu^e + n\mu^i} = \frac{(2n+1)\mu^e}{n(\mu^e + \mu^i) + \mu^e}. \tag{3.56}$$

Using the definition of k given in (3.51) and (3.52), (3.56) can be put in the form

$$\frac{(2n+1)\mu^e}{(n+1)\mu^e + n\mu^i} = \frac{k(2n-1)}{n+k} \tag{3.57}$$

$$= 2k + \frac{k(1-2k)}{n+k}. \tag{3.58}$$

The final form for C_n then becomes

$$C_n = \left[2k + \frac{k(1-2k)}{n+k}\right]A_n. \tag{3.59}$$

Substitution of the coefficients B_n and C_n into (3.46) and (3.47) then yields the infinite series solutions given in (3.42) and (3.43) for the MM model. It is possible to derive closed form solutions by summing up all the terms in the infinite series which will then lead to the closed form solutions given in (3.44) and (3.45). The proof is as follows. The second term on the right side of (3.42) can be represented (using (3.40)) as

$$-\sum_{n=0}^{\infty}(1-2k)\frac{a^{2n+1}}{r^{n+1}}A_nS_n(\theta,\phi) = -(1-2k)\frac{a}{r}\Phi_0(\frac{a^2}{r},\theta,\phi) \tag{3.60}$$

In order to express the last term in (3.42), we observe that

$$R^{k-1}\Phi_0(R,\theta,\phi) = R^{k-1}\sum_{n=0}^{\infty}A_nR^nS_n(\theta,\phi) \tag{3.61}$$

Integrating the above expression from 0 to $\frac{a^2}{r}$, we get

$$\int_0^{a^2/r} R^{k-1}\Phi_0(R,\theta,\phi)\,dR = \sum_{n=0}^{\infty}[\frac{1}{n+k}R^{n+k}A_nS_n(\theta,\phi)]|_0^{\frac{a^2}{r}} \tag{3.62}$$

Substituting the limits, one gets

$$\int_0^{a^2/r} R^{k-1}\Phi_0(R,\theta,\phi)\,dR = \sum_{n=0}^{\infty}[\frac{1}{n+k}(\frac{a^2}{r})^{n+k}A_nS_n(\theta,\phi)] \tag{3.63}$$

The above equation can be re-written as

$$\int_0^{a^2/r} R^{k-1}\Phi_0(R,\theta,\phi)\,dR = \frac{a^{2k}}{r^k}\sum_{n=0}^{\infty}[\frac{1}{n+k}\frac{a^{2n}}{r^n}A_nS_n(\theta,\phi)] \tag{3.64}$$

which can be recast into the form

$$\frac{r^k}{a^{2k}}\int_0^{a^2/r} R^{k-1}\Phi_0(R,\theta,\phi)\,dR = \sum_{n=0}^{\infty}[\frac{1}{n+k}\frac{a^{2n}}{r^n}A_nS_n(\theta,\phi)] \tag{3.65}$$

Substitution of (3.60) and (3.65) into (3.42) yields the closed form solution for Φ^e given in (3.44). Thus the infinite series solution for Φ^e is represented in a closed form using the summation of the series. Note that the exact solution for $\Phi^e(r,\theta,\phi)$ contains a definite integral that depends on the value of the permeability parameter k. The mixed boundary condition applied on the surface of the spherical boundary is the cause for such a dependence. In a similar fashion the potential in the interior phase can be written in a closed form. Re-writing (3.43) in the form

$$\Phi^i(r,\theta,\phi) = \sum_{n=0}^{\infty} 2kr^n A_n S_n(\theta,\phi) + \sum_{n=0}^{\infty} \frac{k(1-2k)}{n+k} r^n A_n S_n(\theta,\phi) \tag{3.66}$$

$$= 2k\sum_{n=0}^{\infty} r^n A_n S_n(\theta,\phi) + k(1-2k)\sum_{n=0}^{\infty} \frac{1}{n+k} r^n A_n S_n(\theta,\phi) \tag{3.67}$$

Using the approach given for the exterior phase potential (see above), the expression (3.67) can be written in closed form leading to the closed form solution given in (3.45). This completes the proof of the the Theorem 4.1 for the MM model.

Therefore, the Maxwell-Maxwell model admits exact solutions for the exterior and interior phases involving a magnetized sphere. As mentioned in section 2.1, the exact solutions given in (3.44) and (3.45) contain several special cases. In the following subsection we record the results for the special case $k = 1$.

3.1.1 Ideal superconducting sphere in an external field ($k = 1$)

Since $k = \frac{\mu^e}{\mu^e+\mu^i}$, $\mu^i = 0$ gives $k = 1$. For this choice of k, the physical problem represents that of an ideal superconducting sphere placed in a given external magnetic field Φ_0 [22, 26]. In this case, Φ^i is irrelevant and Φ^e becomes

$$\Phi^e(r,\theta,\phi) = \Phi_0(r,\theta,\phi) + \frac{a}{r}\Phi_0(\frac{a^2}{r},\theta,\phi) - \frac{1}{a}\int_0^{\frac{a^2}{r}} \Phi_0(R,\theta,\phi)\, dR \tag{3.68}$$

For $k = 1$, the expression (3.68) is the same an given in [26]. Interestingly, the physical problem is the same as that of a flow of an incompressible inviscid fluid around a fixed sphere [22]. Mathematically, the problem reduces to a Neumann boundary value problem for the Laplace equation involving a spherical boundary. The other special cases can be explored in a similar fashion.

Expressions for some magnetic induction fields in the absence of a sphere are given in Table 3.1. In the following sections we use (3.44)-(3.45) to discuss several illustrative examples in order to justify our unique approach for the MM model. In particular, we derive analytic solutions for

Fields	$\Phi_0(r,\theta,\phi)$	B_{0x}	B_{0y}	B_{0z}
Constant	$Hr\cos\theta$	0	0	H
Linear field	$\frac{H_{11}}{3}r^2 S_2(\theta,\phi)$	$\frac{4}{3}H_{11}x$	$\frac{-2}{3}H_{11}y$	$\frac{-2}{3}H_{11}z$
Magnetic pole	$\frac{m}{R_1}$	$\frac{-mx}{R_1}$	$\frac{-my}{R_1}$	$\frac{-m(z-c)}{R_1}$
Magnetic Dipole	$\frac{M_1 r\sin\theta\cos\phi}{R_1^3}$	$M_1(\frac{1}{R_1^3}-\frac{3x^2}{R_1^5})$	$\frac{-3M_1xy}{R_1^5}$	$\frac{-3M_1x(2-c)}{R_1^5}$
(along x - direction)				

Table 3.1
Expressions for Φ_0, and its components B_{0x} , B_{0y} , and B_{0z} in cartesian coordinates generated by some magnetostatic fields. Here H, H_{11}, m, M_1 are constants and $R_1^2 = x^2+y^2+(z-c)^2$, and $S_2(\theta,\phi) = -\frac{1}{2}(3\cos^2\theta - 1) + \frac{3}{2}\sin^2\theta\cos 2\phi$.

various externally imposed potentials in the presence of a permeable sphere. We also make an attempt to interpret the image systems from our analytic solution.

3.2 Constant Magnetic Induction

Let us consider a constant magnetic field of strength H applied in the z-direction. The magnetic induction vector $\mathbf{B}$ for this field is $\mathbf{B}_0 = H\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ is the unit vector in z-direction. The corresponding expression for the magnetostatic potential function in the absence of any boundaries is given by

$$\Phi_0(r,\theta,\phi) = Hr\cos\theta \tag{3.69}$$

If the magnetized sphere of radius $r=a$ is introduced into this constant magnetic induction field, the modified potentials in the exterior and interior phases using (3.44)-(3.45) become

$$\begin{aligned}
\Phi^e(r,\theta,\phi) &= H\cos\theta\left[r-(1-2k)\frac{a}{r}\frac{a^2}{r}+k(1-2k)\frac{r^{k-1}}{a^{2k-1}}\int_0^{\frac{a^2}{r}} R^k dR\right] \\
&= H\cos\theta\left[r-\frac{a^3}{r^2}+k(1-2k)\frac{r^{k-1}}{a^{2k-1}}\frac{a^{2k+2}}{(k+1)r^{k+1}}\right] \\
&= H\quad\left[r\cos\theta-\left(\frac{1-2k}{k+1}\right)\frac{a^3}{r^2}\cos\theta\right]
\end{aligned} \tag{3.70}$$

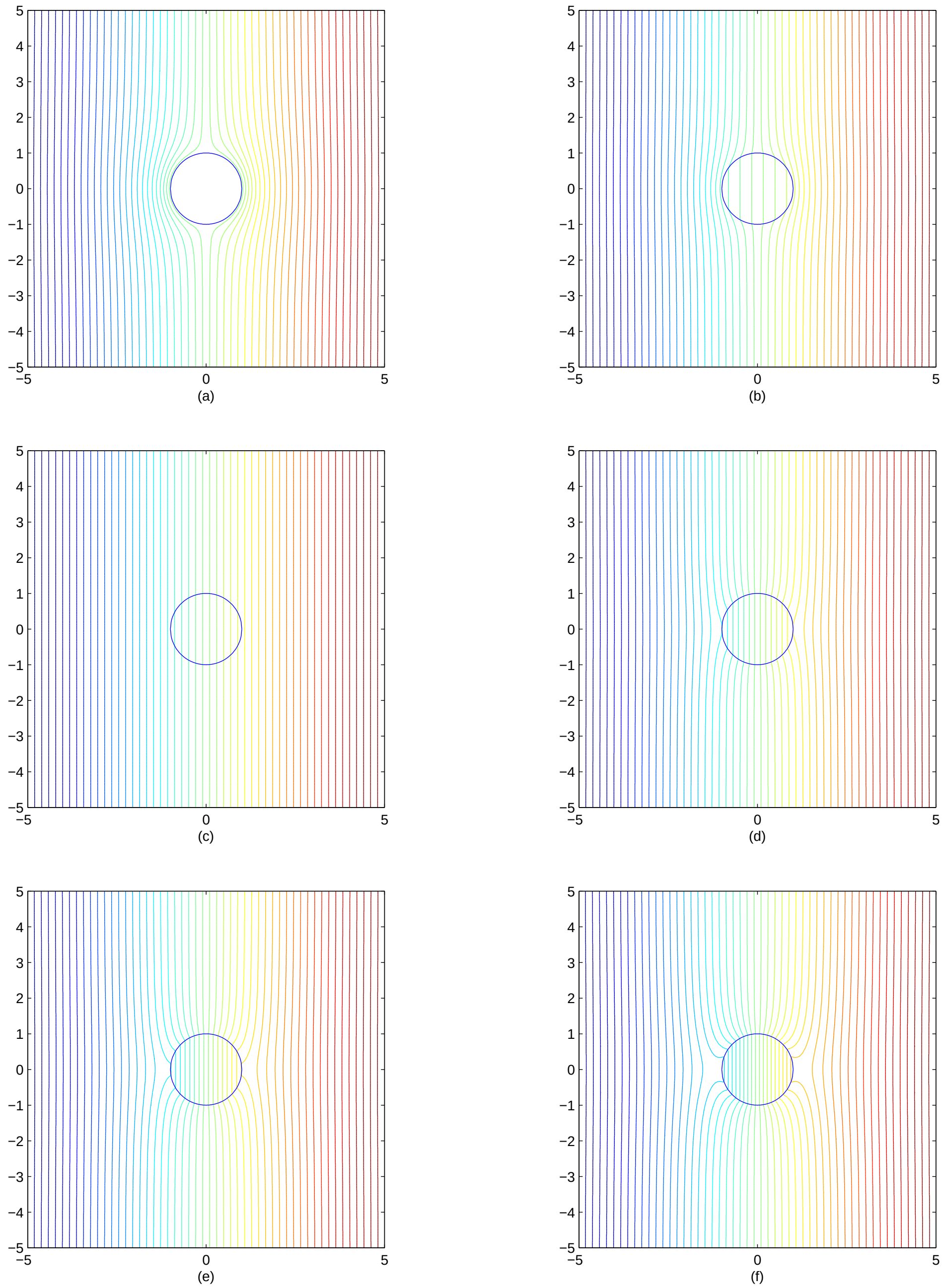

Figure 3.4
Potential plots for a sphere of radius 1 in a constant magnetic field using the MM model for various of k: (a) $k = 0$; (b) $k = 0.25$; (c) $k = 0.5$; (d) $k = 0.75$; (e) $k = 1$; (f) $k = 1.5$ (unphysical).

Similarly

$$\Phi^i(r,\theta,\phi) = H \quad \left[\frac{3kr\cos\theta}{k+1}\right] \tag{3.71}$$

The image system in the exterior phase for constant magnetic induction consists of a dipole of strength $\frac{-(1-2k)}{k+1}a^3$ located at the center of the sphere. Note that the strength of the image dipole depends or the parameter k and the radius a of the sphere. For $k < \frac{1}{2}$, the sign of the dipole is negative, and for $k > \frac{1}{2}$ the sign is positive. For $k = \frac{1}{2}$, the image dipole vanishes. In this case the magnetic permeabilities of the exterior and interior phases coincide.

The potential plots for a magnetized sphere placed in a constant external magnetic field are shown in Figure 3.4 for various value of the permeability parameter k. The contours are drawn in a plane parallel to the xy-plane. When $k = 0$, the situation refers to that of a Dirichlet problem for a sphere in electrostatics [10, 7]. The field lines coming from positive infinity go around the sphere and reach negative infinity (see Figure. 3.4(a)). For $k > 0$, the field lines in both phases exist as seen from Figure 3.4$(b)-(f)$. Two patterns are seen for different ranges of k. For $k < 0.5$, the potential field in the external phase goes around the sphere as in the case of $k = 0$, (Figure 3.4$(b)-(c)$). However, when $k > 0.5$, the field lines in the exterior region bend towards the magnetized sphere and then move away after hitting the spherical surface as shown in Figure 3.4$(d)-(f)$. The interior field lines get denser for $k \geq 0.5$. We remark that when $k > 1$, the problem may represent an unphysical situation since μ^i is *negative* in this case.

3.3 Linear Magnetic Induction

Next we consider a linear magnetic induction field in three-dimensional space. The magnetic field vector $\mathbf{B}_0 = < B_{0x}, B_{0y}, B_{0z} >$ for this field has cartesian components given by

$$B_{0x} = \frac{4}{3}H_{11}x, \qquad B_{0y} = \frac{-2}{3}H_{11}y, \qquad B_{0z} = \frac{-2}{3}H_{11}z \tag{3.72}$$

Using $\mathbf{B}_0 = -\nabla\Phi_0$, the magnetic scalar potential is

$$\Phi_0(x,y,z) = \frac{H_{11}}{3}(2x^2 - y^2 - z^2) \tag{3.73}$$

In spherical polar coordinates Φ_0 becomes

$$\Phi_0(r,\theta,\phi) = \frac{H_{11}}{3} r^2 \left[2\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi - \cos^2\theta \right]$$

which can be written as

$$\Phi_0(r,\theta,\phi) = \frac{H_{11}}{3} r^2 \left[-\frac{1}{2}(3\cos^2\theta - 1) + \frac{3}{2}\sin^2\theta\cos 2\phi \right]$$

We introduce the magnetized sphere into this field given in (3.73). The modified potentials are found using (3.44) and (3.45) as

$$\Phi^e(r,\theta,\phi) = \frac{H_{11}}{3} \left[1 - \frac{2(1-2k)}{k+2} \frac{a^5}{r^5} \right] r^2 S_2(\theta,\phi) \tag{3.74}$$

$$\Phi^i(r,\theta,\phi) = \frac{H_{11}}{3} \left[\frac{5k}{k+2} \right] r^2 S_2(\theta,\phi) \tag{3.75}$$

where

$$S_2(\theta,\phi) = -P_2(\cos\theta) + \frac{1}{2} P_2^2(\cos\theta)\cos 2\phi. \tag{3.76}$$

Here the Legendre polynomials are given by

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) \tag{3.77}$$

$$P_2^2(\cos\theta) = 3\sin^2\theta \tag{3.78}$$

In the present case the image system in the exterior phase consists of a magnetic quadrupole of strength $\frac{-2(1-2k)}{k+2}a^3$ located at the center of the sphere. The direction of the quadrupole changes from $k > \frac{1}{2}$ to $k < \frac{1}{2}$ and vanishes when $k = \frac{1}{2}$.

The level curves for the potential functions in the exterior and interior phases are plotted in Figure 3.5, for various k. Generally, the field lines are symmetrical in the xy-plane. When $k = 0$, that is, $\mu^i \to \infty$, the field lines represent those in the context of electrostatics as shown in Figure 3.5(a). As k increases from zero, the field lines start appearing in both phases as can be seen in Figure 3.5(b)-(f). The field contours are also symmetrical in the interior phase when $k > 0.5$. The

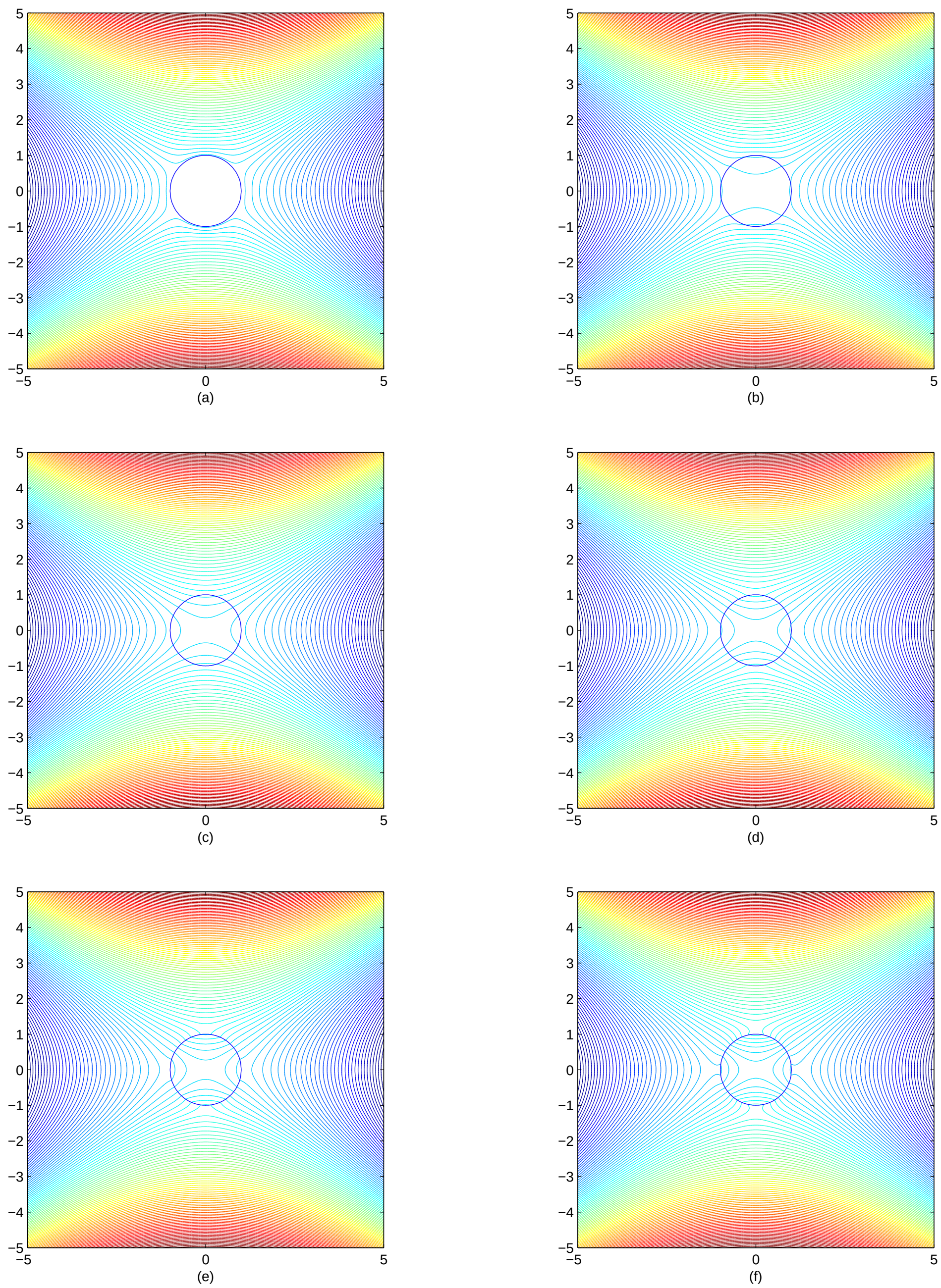

Figure 3.5
Potential plots for a sphere of radius 1 in a linear magnetic field using Maxwell-Maxwell model for various values of k: (a) $k = 0$; (b) $k = 0.25$; (c) $k = 0.5$; (d) $k = 0.75$; (e) $k = 1$; (f) $k = 1.5$ (unphysical).

potential field lines inside the sphere get increasingly closer. It appears that the field lines cross at the center of the magnetized sphere. This scenario seems to exist for all $k > 0$.

We remark that there are five other combinations of x, y, z that lead to linear magnetic induction fields. Their linear combination can be written as

$$\begin{aligned} \Phi_0(r,\theta,\phi) &= \frac{H_{22}}{3}(-x^2+2y^2-z^2) \\ &+ \frac{H_{33}}{3}(-x^2-y^2+2z^2) \\ &+ (H_{12}+H_{21})xy+(H_{13}+H_{31})xz \\ &+ (H_{23}+H_{32})yz. \end{aligned} \tag{3.79}$$

In the above equation H_{ij} are constants. Note that each term in the above expression is a solution of the Laplace equation. The analytic solutions for these other linear fields can be obtained in a similar way as explained in the preceding paragraphs. The image system in each case will have a dipole at the sphere center whose strength depends on the radius a and the permeability parameter k.

3.4 Magnetic Pole at (0,0,c)

Let us now consider a magnetic pole (Magnetic Source) of strength m located at (0,0,c), $c > a$. The unperturbed potential $\Phi_0(x,y,z)$ can be obtained by solving

$$\nabla^2\Phi_0 = m\delta(x,y,z-c) \tag{3.80}$$

where $\delta(x,y,z-c)$ is the Dirac-Delta function. The solution of the Poisson equation [10, 7] is

$$\Phi_0(r,\theta,\phi) = \frac{m}{\mu^e}\frac{1}{R_1} \tag{3.81}$$

where $R_1^2 = r^2 - 2cr\cos\theta + c^2$ (see Table 3.1). If we introduce the magnetized sphere of radius $r = a$ into this unperturbed field with a magnetic source, the modified potentials in the two phases can be constructed using (3.44) and (3.45). Using that

$$\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right) = \frac{m}{\mu^e}\frac{a}{r}\frac{1}{\sqrt{\frac{a^4}{r^2}-2c\frac{a^2}{r}\cos\theta+c^2}} \tag{3.82}$$

$$= \frac{m}{\mu^e}\frac{a}{r}\frac{r}{c\sqrt{r^2-2\frac{a^2}{c}r\cos\theta+\frac{a^4}{c^2}}} \tag{3.83}$$

$$= \frac{m}{\mu^e}\frac{a}{cR_2} \tag{3.84}$$

where $R_2^2 = r^2 - 2\frac{a^2}{c}r\cos\theta + \frac{a^4}{c^2}$. We also use that (for the purpose of representing the integrand)

$$\frac{a}{r}\Phi_0(R,\theta,\phi) = \frac{m}{\mu^e}\frac{a}{r}\frac{1}{\sqrt{R^2-2cR\cos\theta+c^2}}. \tag{3.85}$$

Thus the potentials of the perturbed field in the two phases become

$$\begin{aligned}\Phi^e &= \frac{m}{\mu^e}\left[\frac{1}{R_1}-(1-2k)\frac{a}{cR_2}\right.\\ &+ \left.\left(\frac{a}{r}\right)k(1-2k)\left(\frac{r}{a^2}\right)^k\int_0^{a^2/r}\frac{R^{k-1}}{\sqrt{R^2-2cR\cos\theta+c^2}}\,dR\right]\end{aligned} \tag{3.86}$$

$$\Phi^i = \frac{m}{\mu^e}\left[\frac{2k}{R_1}+k(1-2k)\int_0^r\frac{R^{k-1}}{\sqrt{R^2-2cR\cos\theta+c^2}}\,dR\right] \tag{3.87}$$

The last term in (3.86) can be re-written in the following form

$$k(1-2k)\frac{a}{r}\left(\frac{r}{a^2}\right)^k\int_0^{a^2/r}\frac{R^{k-1}}{\sqrt{R^2-2cR\cos\theta+c^2}}\,dR \tag{3.88}$$

$$= k(1-2k)a^{1-2k}c^{k-1}\int_0^{a^2/r}\frac{R^{k-1}}{\sqrt{R^2-2cR\cos\theta+c^2}}\,dR \tag{3.89}$$

$$= k(1-2k)\frac{a^{1-2k}}{c^{1-k}}\int_0^{a^2/r}\frac{1}{\sqrt{R^2-2cR\cos\theta+c^2}}R^{-(k-1)}\,dR \tag{3.90}$$

Now we can write (3.86) in the form

$$\Phi^e(r,\theta,\phi) = \frac{m}{\mu_1}\left[\frac{1}{R_1} - (1-2k)\frac{a}{cR_2} + k(1-2k)\frac{a^{1-2k}}{c^{1-k}}\int_0^{a^2/r} \frac{1}{\sqrt{R^2 - 2cR\cos\theta + c^2}} R^{-(1-k)}\, dR\right] \qquad (3.91)$$

The image system in the exterior phase can be interpreted as follows. It consists of

- a pole of strength $-(1-2k)\frac{a}{c}m$ at the inverse point $(0,0,\frac{a^2}{c})$
- a line distribution of poles from the center of the phase $(0,0,0)$ to the inverse point $(0,0,\frac{a^2}{c})$.

The strength of the distribution is given by $k(1-2k)\frac{a^{(1-2k)}}{(cR)^{1-k}}m$. The image system agrees with that found by Carl Neumann (see [10]).

3.5 Forces exerted on a sphere for the Maxwell-Maxwell model

It is of interest to calculate the forces acting on a magnetized sphere placed in the field of a magnetic pole or a dipole. In the content of superconductivity such forces are termed levitation forces [22]. The levitation forces exerted on the superconducting sphere due to poles and dipoles have been calculated by many researchers [28, 22, 4]. The levitation force due to a circular current loop is also found in [4]. Recently, Trombley an Palaniappan [26] calculated the force acting on the superconducting sphere in the field of a straight line current. They found that the force can be expressed in an integral form involving a logarithmic function. They stated that the integral can be evaluated in terms of special functions. In the following, we sketch the calculation of the force acting on the sphere due to a magnetic pole and in the next section (section 3.6) we determine the force acting on a sphere in the presence of a point dipole.

3.5.1 Force due to a magnetic pole

As in [22], the force acting on the sphere of radius a due to a magnetic pole of strength m located at $(x,y,z)=(0,0,c)$ where $c>a$ is given by

$$\mathbf{F} = -m[\nabla(\Phi^e - \Phi_0)]|_{r=c} \tag{3.92}$$

Here Φ^e is the exterior potential and Φ_0 is the potential due to a pole in the absence of a sphere. The expressions for Φ^e and Φ_0 are already given in section 3.4 (see equation (3.81) and (3.91). From (3.81) and (3.91) we see that $\Phi^e - \Phi_0$ can be written in a convenient form as

$$\Phi^e - \Phi_0 = \frac{m}{\mu^e}(1-2k)\frac{a}{c}\left[-\left(r-\frac{a^2}{c}\right)^{-1} + k\int_0^1 \left(r - \frac{\gamma a^2}{c}\right)^{-1} \gamma^{-(1-k)}\, d\gamma\right] \tag{3.93}$$

where γ here is the dummy (integration) variable. Substitution of (3.93) into (3.92) yields the force given by

$$\mathbf{F} = \frac{m^2}{\mu^e}(1-2k)\frac{a}{c}\left[\left(c-\frac{a^2}{c}\right)^{-2} - k\int_0^1 \left(c - \frac{\gamma a^2}{c}\right)^{-1} \gamma^{-(1-k)}\, d\gamma\right]\hat{\mathbf{e}}_z \tag{3.94}$$

We observe that the force due to a magnetic pole depends on

- the radius of the sphere a
- the location of the initial pole c
- the magnetic permeability parameter k

Further, the force contains an integral that can be expressed in terms of hypergeometric functions [10, 7]. We note that when $k=1$, the integral can be evaluated easily and the force in this case becomes

$$\mathbf{F} = \frac{m^2}{\mu^e}\frac{a^3}{c(c^2-a^2)^2}\hat{\mathbf{e}}_z \tag{3.95}$$

The above expression for the force agrees with that found in [11] for the case of a superconducting sphere. We see that in the limit of $c >> a$, that is when the pole is located far away from the sphere, the force becomes approximately

$$\mathbf{F} = \frac{m^2}{\mu^e}\frac{a^3}{c^5}\hat{\mathbf{e}}_z \tag{3.96}$$

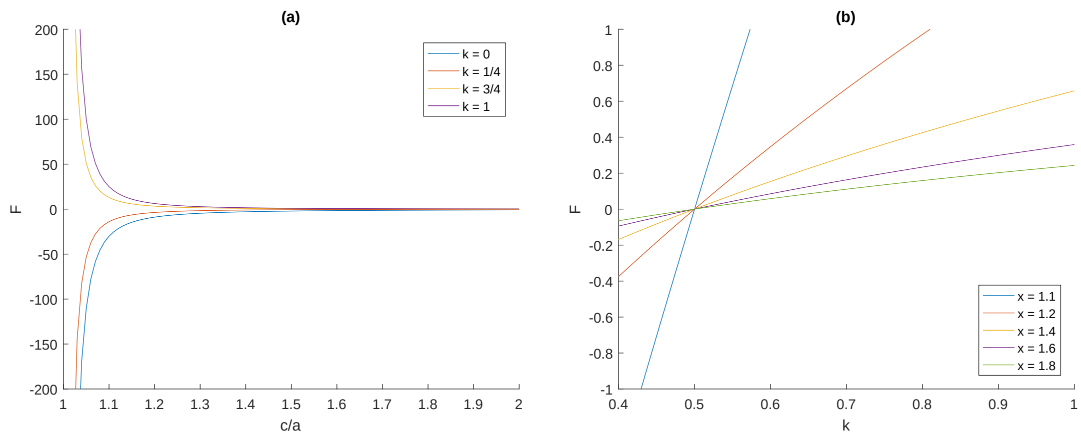

Figure 3.6
Force on sphere due to a magnetic pole at $(x,y,z) = (0,0,c)$, $c > a$. (a) Force versus location ratio $\frac{c}{a}$ for fixed k; (b) Force versus k for fixed $x = \frac{c}{a}$

In the general case, since $k = \frac{\mu^e}{\mu^e+\mu^i}$ and $1-2k = \frac{\mu^i-\mu^e}{\mu^e+\mu^i}$ we see from (3.94) that the force is

- positive for $k > \frac{1}{2}$ or $\mu^e > \mu^i$
- negative for $k < \frac{1}{2}$ or $\mu^e < \mu^i$

respectively. In other words, the force is attractive for $k < \frac{1}{2}$ and repulsive for $k > \frac{1}{2}$. The force component is plotted versus k for various monopole locations in Figure 3.6(b). It shows a linear variation for each location ratio $\frac{c}{a}$.

3.6 Magnetic dipole at (0,0,c)

We now consider the effect of a sphere on the field of a magnetic dipole. A magnetic dipole is a pair of poles/sources as they shrink to a point while keeping the magnetic moment constant. Two cases are to be considered : (i) A magnetic dipole of strength M_3 in the direction of the z-axis (*radial or axisymmetric*) and (ii) A magnetic dipole of strength M_1 or M_2 in the direction of x or y-axis (*transverse or asymmetric*). We treat these two cases separately in the following

(i) *Radial dipole:*

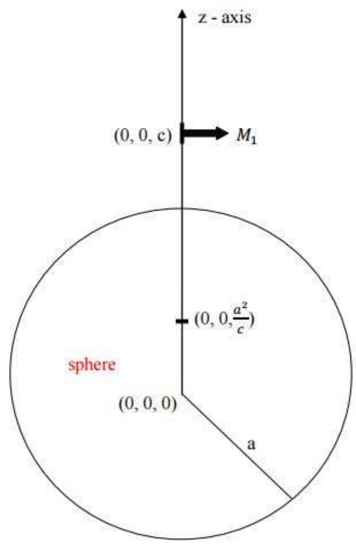

Figure 3.7
Radial dipole

Let the dipole of strength M_3 along the z-direction located at $(0,0,c)$, for $c > a$. The potential function $\Phi_0(r,\theta,\phi)$ in the absence of the sphere is

$$\Phi_0(r,\theta,\phi) = M_3 \tfrac{r\cos\theta - c}{R_1^3}$$

where as before $R_1^2 = r^2 - 2cr\cos\theta + c^2$.

If we introduce the magnetized sphere of radius a centered at $(0,0,0)$ into this dipole field, then the modified potential functions, for the two phases become, calculated using (3.44) and (3.45)

$$\begin{aligned} \Phi^e(r,\theta,\phi) &= M_3 \frac{r\cos\theta - c}{R_1^3} + M_3(1-2k)\left[\frac{-a}{c^2 R_2} - \frac{a^3}{c^3}\frac{(r\cos\theta - \frac{a^2}{c})}{R_2^3}\right] \\ &+ M_3 k(1-2k)c^{k-1}a^{1-2k}\int_0^{a^2/r} R^{k-1}\frac{(R\cos\theta - c)}{R_1^3}\,dR \end{aligned} \tag{3.97}$$

$$\Phi^i(r,\theta,\phi) = M_3\left[2k\frac{(r\cos\theta - c)}{R_1^3} + k(1-2k)\int_0^r R^{k-1}\frac{(R\cos\theta - c)}{R_1^3}\,dR\right] \tag{3.98}$$

where $R_2^2 = r^2 - 2\frac{a^2}{c}r\cos\theta + \frac{a^4}{c^2}$. The first term on the right side of (3.97) is the initial dipole and the remaining terms represent the image system in the magnetized sphere for this radial dipole. The image system for the exterior phase can be interpreted as follows.

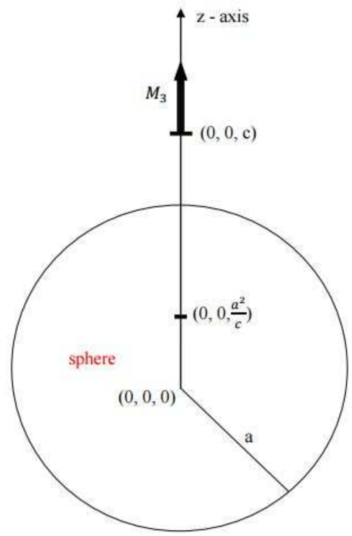

Figure 3.8
Transverse dipole

- The second term on the right side of (3.97) is a source/sink of strength $-M_3\frac{a}{c^2}(1-2k)$ at the inverse point $(0,0,\frac{a^2}{c})$
- The third term on the right side of (3.97) represents a magnetic dipole $-M_3\frac{a^3}{c^3}(1-2k)$
- The last term can be interpreted as the line distribution of magnetic sources and dipoles of strength $-M_3k(1-2k)a^{1-2k}c^{k-1}$. The dipoles are oriented along the radial direction.

When $k=1$ the above image system reduces to the given in [22]. When $k=0$ we recover the electrostatic image of a dipole in a sphere [10].

(ii) *Transverse dipole:*

Let the transverse dipole of strength M_1 (or M_2) along x (or y) direction be located at $(0,0,c)$, where $c>a$. The original potential function $\Phi_0(r,\theta,\phi)$ without the sphere is

$$\Phi_0(r,\theta,\phi) = M_1\frac{r\sin\theta\cos\phi}{R_1^3} \tag{3.99}$$

The expression for the transverse dipole along the y-direction can be written in a similar fashion. If we now introduce the magnetized sphere in to this transverse dipole field, the modified potentials in the two phases become

$$\begin{aligned}\Phi^e(r,\theta,\phi) &= M_1\frac{r\sin\theta\cos\phi}{R_1^3}+M_1(1-2k)\left[\frac{a^3}{c^3}\frac{r\sin\theta\cos\phi}{R_1^3}\right] \\ &+ M_1k(1-2k)c^{k-1}a^{1-2k}\int_0^{a^2/r}R^{k-1}\frac{(R\sin\theta\cos\phi)}{R_1^3}\,dR \qquad (3.100)\end{aligned}$$

and

$$\Phi^i(r,\theta,\phi)=M_1\left[2k\frac{(r\sin\theta\cos\phi)}{R_1^3}+k(1-2k)\int_0^{r}R^{k-1}\frac{(Rr\sin\theta\cos\phi)}{R_1^3}\,dR\right] \qquad (3.101)$$

The first term on the right side of (3.100) is the initial transverse dipole and the remaining terms represent the image system in the magnetized sphere. The image system for a transverse dipole in the exterior phase can be interpreted in the following manner

- The second term is the transverse dipole of strength $M_1\frac{a^3}{c^3}(1-2k)$ in the same direction of the initial magnetic dipole, but located at $(0,0,\frac{a^2}{c})$
- The last term can be interpreted as a combination of line distribution of dipoles and magnetic sources. The total strength of the line dipole distribution is found to be $M_1k(1-2k)a^{1-2k}c^{k-1}$
- If we let $k=1$, we obtain the image system a dipole in the hydrodynamical case [29]. When $k=0$, the electrostatic image of a transverse dipole situated in front of a sphere is recovered

In summary

- The image system for a transverse dipole along the y-direction located outside a magnetized sphere can be found in the same manner.
- By superposition of all the three cases (that is the solutions for a dipole in x, y, and z directions) one can obtain the exterior and interior potentials for a *general dipole* positioned in front of a magnetized sphere.

- It can be seen that the image system for a general dipole consists of a dipole at the inverse point, a line distribution of sources from the origin to the inverse point and a line distribution of dipoles from the origin to the inverse point.

The exact solutions in the forms presented above for a radial and transverse dipole do not seem to be available in the literature. The analytic solutions for Φ^e and $\Phi^i$to any other given potential field $\Phi_0(r, \theta, \phi)$ can be constructed using (3.44) and (3.45).

3.6.1 Force due to a magnetic dipole

The analytic solution for the exterior and interior potentials due to a dipole located at $(x, y, z) = (0, 0, c)$ are derived in this section. Now we derive the expressions for the force due to a dipole acting on a permeable sphere. As said earlier there are two cases namely (i) a radial dipole, and (ii) a transverse dipole. Below we record the relevant expressions calculated using(3.97) and (3.100). The force due to a dipole is determined from [11, 29]

$$\mathbf{F} = [(\mathbf{M} \cdot \nabla)\nabla(\Phi^e - \Phi_0)]|_{r=c} \tag{3.102}$$

where $\mathbf{M}$ represents the magnitude and direction of the dipole and $|_{r=c}$ indicates the quantities to be evaluated at $r = c$, the location of the dipole. For a radial dipole at $(0, 0, c)$, the expression for the force, calculated using (3.102), is

$$\begin{aligned} \mathbf{F} &= \frac{12(1-2k)}{\mu^e} M_3^2 \frac{a^3}{c^7} \Bigg[-\int_0^1 (1-\gamma\frac{a^2}{c^2})^{-4} \gamma^k \, d\gamma \\ &+ 2\int_0^1 (1-\gamma\frac{a^2}{c^2})^{-5} \gamma^k \, d\gamma \Bigg] \hat{\mathbf{e}}_z \end{aligned} \tag{3.103}$$

As seen from (3.103), the force acts along the z-direction and depends on the radius of the sphere, location the initial dipole c, and the magnetic permeability parameter k. The two integrals in (3.103) can be evaluated in terms of hypergeometric functions. But for numerical purposes the integrals are convenient and so we will retain them in their present form. For $k = 1$, the expression for the force, after evaluating the integrals, becomes

$$\mathbf{F} = 6M_3^2 \frac{a^3 c}{(c^2 - a^2)^4} \hat{\mathbf{e}}_z \tag{3.104}$$

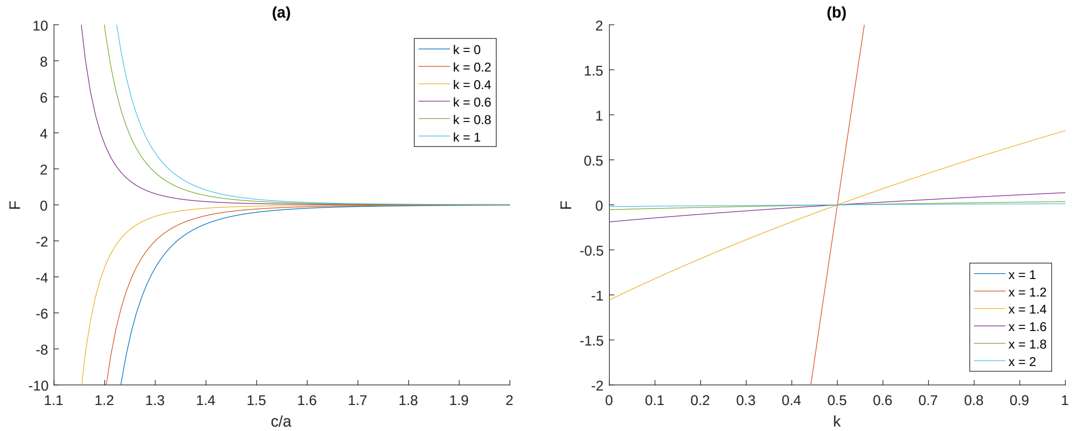

Figure 3.9
Force on sphere due to a radial magnetic dipole at $(0,0,c)$, $c > a$. (a) Force versus location ratio $\frac{c}{a}$ for fixed k; (b) Force versus k for fixed $x = \frac{c}{a}$

This result is the same as that derived in [22] for the force due to a radial dipole acting on a superconducting sphere. The normalized force versus the permeability parameter k shows a linear behavior as seen in Figure 3.9(b). The force due to a transverse dipole can be calculated in a similar fashion. Substitution of (3.100) in (3.102) yields

$$\begin{aligned} \mathbf{F} &= \frac{3(1-2k)}{\mu_e} M_1^2 \frac{a^3}{c^7}\left[-3\int_0^1 \left(1-\gamma\frac{a^2}{c^2}\right)^{-4} \gamma^k \, d\gamma \right. \\ &+ \left. 4\int_0^1 \left(1-\gamma\frac{a^2}{c^2}\right)^{-5} \gamma^k \, d\gamma\right] \hat{\mathbf{e}}_z \end{aligned} \tag{3.105}$$

The force for transverse dipole acts in the z-direction. As before, the parameters a, c, and k dictate the force. The integrals may be evaluated in terms of hypergeometric functions. For $k = 1$, the integrals, can be evaluated in terms of elementary functions and the force becomes

$$\mathbf{F} = 3M_1^2 \frac{a^3 c}{(c^2-a^2)^4}\left[\frac{3}{2} + 2\frac{a^2}{c^2} - \frac{1}{2}\frac{a^4}{c^4}\right]\hat{\mathbf{e}}_z \tag{3.106}$$

The above result is the same as that obtained in [22] for the force due to a transverse dipole acting on a superconducting sphere. The normalized force for a tangential dipole versus the permeability parameter k shows a linear behavior for various dipole locations as seen in Figure 3.10(b).

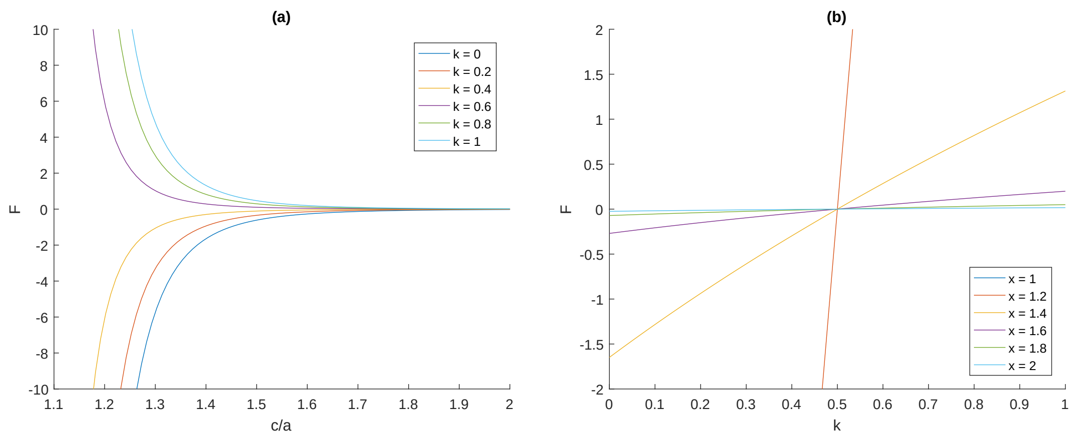

Figure 3.10
Force on sphere due to a transverse magnetic dipole at $(0,0,c)$, $c > a$. (a) Force versus location ratio $\frac{c}{a}$ for fixed k; (b) Force versus k for fixed $x = \frac{c}{a}$

3.7 A new relation for multipole coefficients of the exterior field

The constants B_n given in (3.55) (see section 3.1) can be interpreted as the multipole coefficients for the exterior magnetic field. They play a crucial role in the determination of magnetic permeability factors in a variety of situations [25, 16]. Physically, these coefficients characterize the strength of the poles of different orders (monopole, dipole, quadrupole etc.). In the content of dielectrics, the B_n are known as the Clausius-Massotti factors [9] and their knowledge is crucial for studying properties of dielectric media as explained in [16]. In working with multi-particle dynamics, the calculation of multipole coefficients is a difficult task both analytically and numerically. Any relation or observation regarding the coefficients can be very useful in validating the results observed via analytical and/or numerical computations. Here, we present a new relation on the multipole coefficients B_n . To this end we first define

$$B_{nD} = -A_n a^{2n+1} \tag{3.107}$$

where B_{nD} are the multipole coefficients for the Dirichlet problem for a sphere [19]. Next we take

$$B_{nN} = \frac{n}{n+1} A_n a^{2n+1} \tag{3.108}$$

Here B_{nN} are the multipole coefficients for the Neumann boundary value problem for a sphere [19]. Now the multipole coefficients B_n for our Maxwell-Maxwell model given in equation (3.55) are

$$B_n = [(2k-1) - \frac{k(2k-1)}{n+k}]A_n a^{2n+1} \tag{3.109}$$

The factor in square brackets in the above equation can be written as (using algebra)

$$(2k-1) - \frac{k(2k-1)}{n+k} = -\left[(1-k) - \frac{k(1-k)}{n+k}\right] + \left[k + \frac{k(1-k)}{n+k}\right]\frac{n}{n+1} \tag{3.110}$$

With this decomposition (3.109) takes the form

$$B_n = [(1-k) - \frac{k(1-k)}{n+k}](-A_n a^{2n+1}) + [k + \frac{k(1-k)}{n+k}]\frac{n}{n+1}A_n a^{2n+1} \tag{3.111}$$

We now define the following

$$\Lambda_{nk} = k + \frac{k(1-k)}{n+k}, \qquad 1 - \Lambda_{nk} = (1-k) - \frac{k(1-k)}{n+k} \tag{3.112}$$

Then the multipole coefficients B_n become

$$B_n = (1-\Lambda_{nk})B_{nD} + \Lambda_{nk}B_{nN} \tag{3.113}$$

where B_{nD} and B_{nN} are the Dirichlet and Neumann coefficients given in (3.107) and (3.108). Note that the lower bound for Λ_{nk} is zero and since

$$\Lambda_{nk} = \frac{k(n+1)}{n+k} \tag{3.114}$$

we see that the denominator is $n+k$ greater than the numerator $kn+k$ for $0 \leq k \leq 1$. Therefore, $0 \leq \Lambda_{nk} \leq 1$ unless μ^i is negative. Thus, the multipole relation (3.113) implies that *the multipole coefficients B_n for a mixed boundary value problem are a convex combination of the Dirichlet and the Neumann coefficients.* The constant Λ_{nk} depends on the order of the pole n and the parameter k. We remark that the relation (3.113) is *new* and has not been reported in the literature. As said before, in the determination of potentials with multi-particles, equation (3.113) can be used to check the validity of the results obtained using numerical methods. The constants Λ_{nk} for the fields considered in preceding sections 3.2 and 3.3 are given below.

For a constant field: $\Lambda_{1k} = k + \frac{k(1-k)}{1+k}$

For a linear field: $\Lambda_{2k} = k + \frac{k(1-k)}{2+k}$

For a magnetic pole and dipole the spherical harmonic expansion is of order n and so Λ_{nk} in these cases is similar to (3.112).

In the context of dielectrics, the coefficients B_n given in (3.55) have yet another interpretation. To see this we write (3.55)

$$B_n = \frac{-n(\mu^i - \mu^e)}{n\mu^i + (n+1)\mu^e} a^{2n+1} A_n \tag{3.115}$$

If we take $\mu^i = \varepsilon_1$, and $\mu^e = \varepsilon_2$ and define

$$k_n = \frac{n(\varepsilon_1 - \varepsilon_2)}{n\varepsilon_1 + (n+1)\varepsilon_2} \tag{3.116}$$

Then the factor k_n is the generalization of Clausius-Massotti factor. For $n = 1,2$ the factors k_1 and k_2 are known [9, 10]. This indicates the general validity of Maxwell-Maxwell model, discussed in this chapter. For $n = 1$, the Clausius-Massotti factor is

$$k_1 = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \tag{3.117}$$

which is the same as given in [10]. For $n = 2$ we get

$$k_2 = \frac{2(\varepsilon_1 - \varepsilon_2)}{2\varepsilon_1 + 3\varepsilon_2} \tag{3.118}$$

and so on. By the analogy the generalized Clausius-Massotti factor k_n also possesses a convex combination form as does the multipole coefficients B_n.

In this chapter we consider another popular model in the context of superconductivity in magnetostatics using the Maxwell-London theory. The governing equations for the axisymmetric fields together with the boundary conditions are provided in Section 2.2 (see equations (2.14) - (2.17)). As noted there, the physical situation is that a superconducting sphere is placed in an external axisymmetric magnetic field with a penetration depth of the field defined as λ. Mathematically, the problem reduces to a mixed boundary value problem for the exterior and interior potentials. It should be noted that the exterior potential satisfies the Laplace equation while the interior potential satisfies the Helmholtz equation. We use an infinite series expansion method for the two phases and give the results in the form of a theorem. To this end, we define Ψ_0 to be a given axisymmetric potential in the absence of any boundaries. Hence Ψ_0 does not depend on the azimuthal angle ϕ. Since $\nabla^2\Psi_0 = 0$, we can represent Ψ_0 in the form suggested by [7, 8]

$$\Psi_0(r,\theta) = \sum_{n=1}^{n} A_n r^{n+1} P_n^1(\cos\theta) \tag{4.119}$$

where $P_n^1(\cos\theta)$ is the associated Legendre function of the second kind. We call the magnetic flux density function Ψ as an axisymmetric potential in the sequel. Below we state and prove the theorem representing the solutions of the BVP (2.14) - (2.17) for a superconducting sphere placed in an external axisymmetric magnetic field.

Theorem 4.1: Let $\Psi_0(r,\theta)$ be an arbitrary axisymmetric potential field in the absence of any boundaries. If a superconducting sphere $r = a$ is introduced in the field of an external field Ψ_0, then the modified axisymmetric potentials in the exterior and interior phases for the ML model satisfying the mixed BVP (2.14) - (2.17) are given by

$$\Psi^e(r,\theta) = \sum_{n=1}^{n} \left[A_n r^{n+1} + [\frac{(2n+1)f_n(\lambda a)}{\lambda a f_{n-1}(\lambda a)} - 1] A_n \frac{a^{2n+1}}{r^n} \right] P_n^1(\cos\theta) \tag{4.120}$$

$$\Psi^i(r,\theta) = \sum_{n=0}^{\infty} \frac{(2n+1)A_n a^n}{(\lambda a) f_{n-1}(\lambda a)} f_n(\lambda r) P_n^1(\cos\theta) \tag{4.121}$$

Fields	$\Psi_0(r,\theta)$	B_{0x}	B_{0y}	B_{0z}
Constant	$\frac{H}{2}r^2\sin^2\theta$	0	0	H
Linear field	$H_{11}(r^3\sin^2\theta\cos\theta)$	$H_{11}x$	$H_{11}y$	$-2H_{11}z$
Magnetic pole	$m(1+\frac{r\cos\theta-c}{R_1})$	$\frac{-mx}{R_1}$	$\frac{-my}{R_1}$	$\frac{-m(z-c)}{R_1}$

Table 4.2
Expressions for the potential function Ψ_0, and the field components in cartesian coordinates B_{0x}, B_{0y}, and B_{0z} generated by some magnetostatic fields. Here H, H_{11}, and m are constants and $R_1^2 = x^2+y^2+(z-c)^2$.

Proof: Let $\Psi_0(r,\theta)$ be a given axisymmetric potential (magnetic flux density function) in the absence of any boundary. Then Ψ_0 can be expanded in an infinite series as given in (4.119). When the spherical superconductor is introduced into the field of Ψ_0 the exterior potential, according to the general solutions of axisymmetric Laplace equations given in section 2.4 is

$$\Psi^e(r,\theta) = \sum_{n=1}^{n}\left[A_n r^{n+1} + \frac{B_n}{r^n}\right]P_n^1(\cos\theta) \tag{4.122}$$

In the interior phase, the axisymmetric potential satisfies the axisymmetric Helmholtz equation (2.15) and so a suitable choice for $\Psi^i(r,\theta)$ is

$$\Psi^i(r,\theta) = \sum_{n=0}^{\infty} C_n f_n(\lambda r) P_n^1(\cos\theta) \tag{4.123}$$

where $f_n(\lambda r) = \sqrt{\frac{\pi}{2\lambda r}} I_{n+\frac{1}{2}}(\lambda r)$ is the spherical Bessel function of the first kind [1, 7] and $I_n(\lambda r)$ is the modified Bessel function of the first kind. The constant A_n is associated with the given axisymmetric potential field and we determine B_n and C_n using the boundary conditions.
Using the boundary conditions $\Psi^e = \Psi^i$ on $r=a$, we get

$$\sum_{n=0}^{\infty}\left[C_n f_n(\lambda r) - \frac{B_n}{a^n}\right]P_n^1(\cos\theta) = \sum_{n=0}^{\infty} A_n a^{n+1} P_n^1(\cos\theta). \tag{4.124}$$

Application of the boundary condition $\frac{\partial \Psi^e}{\partial r} = \frac{\partial \Psi^i}{\partial r}$ on $r=a$ yields

$$\sum_{n=0}^{\infty}\left[\lambda C_n f_n'(\lambda a) + n\frac{B_n}{a^{n+1}}\right]P_n^1(\cos\theta) = \sum_{n=0}^{\infty}(n+1)A_n a^n P_n^1(\cos\theta) \tag{4.125}$$

Solving (4.124)-(4.125) we obtain

$$C_n = \frac{(2n+1)A_n a^n}{\lambda a f_{n-1}(\lambda a)} \tag{4.126}$$

In the derivation of C_n we have used the following identities that are found in [7]:

$$\begin{aligned} f_n'(z) &= f_{n-1}(z) - \frac{n-1}{z} f_n(z) && (4.127) \\ f_n'(\lambda a) &= f_{n-1}(\lambda a) - \frac{n-1}{\lambda a} f_n(\lambda a) && (4.128) \end{aligned}$$

Next we use equation (4.126) to find B_n in terms of A_n

$$\sum_{n=0}^{\infty} \left[C_n f_n(\lambda a) - \frac{B_n}{a^{n+1}} - A_n a^n \right] P_n^1(\cos\theta) = 0 \tag{4.129}$$

Hence

$$\begin{aligned} \frac{B_n}{a^{n+1}} &= C_n f_n(\lambda a) - A_n a^n && (4.130) \\ B_n &= (C_n f_n(\lambda a) - A_n a^n) a^{n+1} \quad \text{for all } n \geq 1 && (4.131) \\ B_n &= \left(\left[\frac{(2n+1)A_n a^n}{\lambda a f_{n-1}(\lambda a)} \right] f_n(\lambda a) - A_n a^n \right) a^{n+1} && (4.132) \\ B_n &= \left[\frac{(2n+1) f_n(\lambda a)}{\lambda a f_{n-1}(\lambda a)} - 1 \right] A_n a^{2n+1}. && (4.133) \end{aligned}$$

Thus we have determined all the unknown constants in (4.122) and (4.123). Substitution of the constants yield the axisymmetric potentials in the exterior and interior phases in the presence of superconducting sphere. This completes the proof of the exact solutions given in ((4.120) and (4.121).

Notice that the exact solutions for the Maxwell-London model are in infinite series form. Due to the occurrence of the Bessel function it is not possible to express the solution in closed form. Nevertheless, these infinite series solutions can be used for any axisymmetric magnetic induction field problem involving a superconducting sphere. Below we will use (4.120)-(4.121) to construct analytic solutions for same specific induced fields. In particular, we will derive potentials due to constant and linear magnetic induction in the presence of a superconducting sphere. Expressions for some axisymmetric magnetic fields in terms of Ψ_0 are provided in Table 4.2.

4.1 Constant Magnetic Induction

Let us consider a constant magnetic induction along the z-direction given by $\mathbf{B}_0 = H\hat{\mathbf{e}}_z$. The corresponding axisymmetric potential in spherical coordinates is $\Psi_0(r,\theta) = \frac{H}{2}r^2\sin^2\theta$. This field corresponds to the solution with $n=1$ and $A_1 = \frac{H}{2}$ in (4.119). If we introduce the superconducting sphere into this field, the axisymmetric potentials for the exterior and interior phases, using the theorem 4.1 with $n=1$, become

$$\Psi^e(r,\theta) = \frac{H}{2}r^2\sin^2\theta + \left[\frac{3f_1(\lambda a)}{\lambda a f_0(\lambda a)} - 1\right]H\frac{a^3}{r}\sin^2\theta \tag{4.134}$$

$$\Psi^i(r,\theta) = \left(\frac{3a}{2\lambda a f_0(\lambda a)}\right)f_1(\lambda r)H\sin^2\theta. \tag{4.135}$$

The contour plots for a constant external magnetic potential plotted using (4.134) and (4.135) are shown in Figure 4.11 for various values of the penetration depth parameter $\lambda_1 = \frac{1}{\lambda}$. When λ_1 is small, that is for large λ, the field lines penetrate the spherical conductor as if it nearly was not there as seen from Figure 4.11(a)-(b). As we increase λ_1, the interior lines start to disappear as shown in Figure 4-11(c)-(d). As λ_1, gets larger, that is, when the penetration depth λ is small, the field lines begin to stop penetrating the spherical boundary (See Figure 4.11(e)). For very large λ_1, that is when $\lambda \to 0$, the potential field goes around the sphere as seen in Figure 4.11(f). This is the situation for an ideal superconductor in an external magnetic field. The scenario described here agrees with those predicted analytically by Matute [17]. The image system in the exterior phase consists of a dipole of strength

$$\left[\frac{3f_1(\lambda a)}{\lambda a f_0(\lambda a)} - 1\right]Ha^3$$

located at the center of the sphere. The strength of the image dipole depends on the radius a and the penetration depth λ. Likewise, the potentials in the exterior and interior phases can be constructed for constant magnetic induction along any direction.

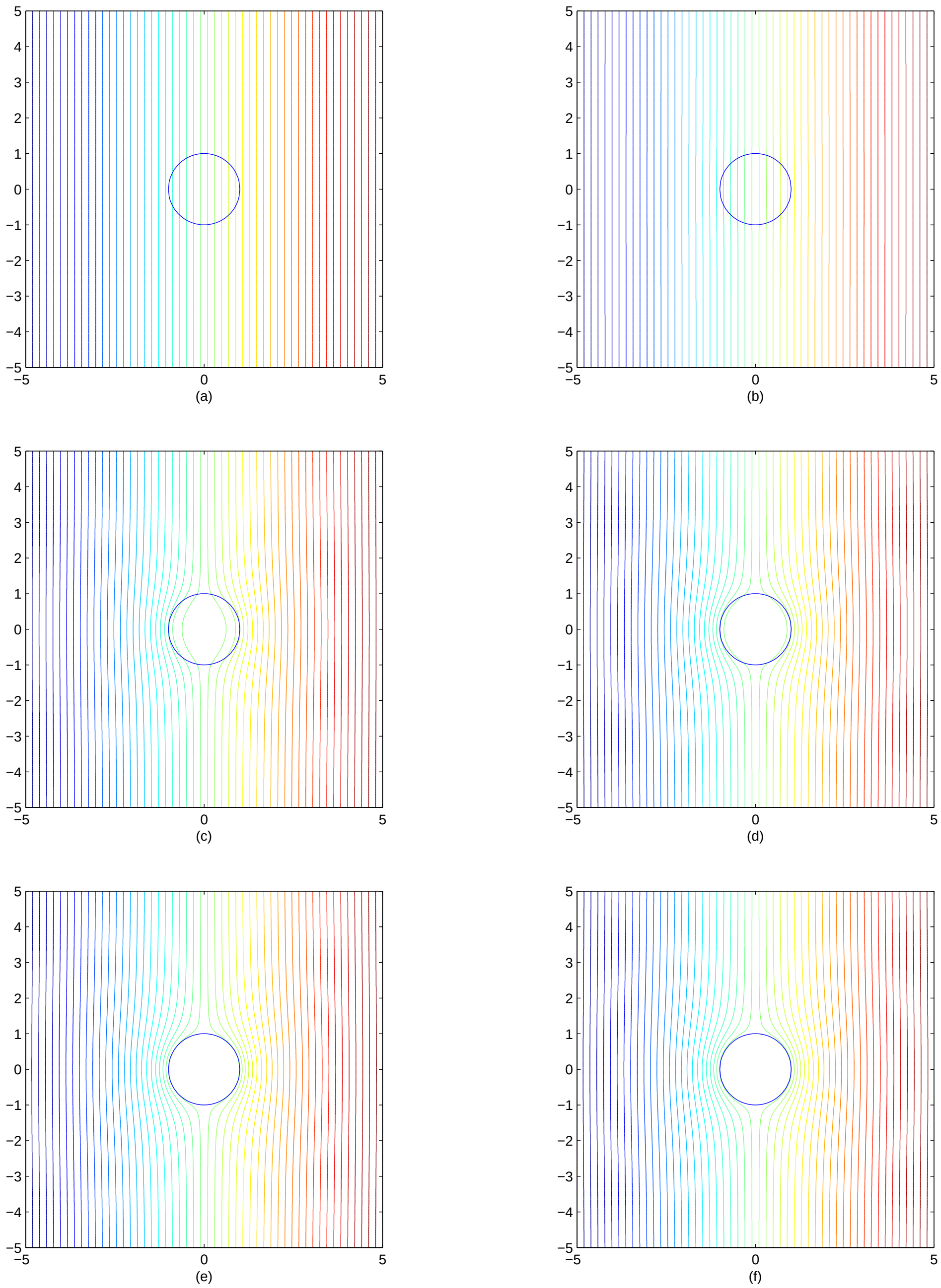

Figure 4.11
Potential plots for a sphere in a constant magnetic field using the Maxwell-London model for various penetration depth λ_1: (a) $\lambda_1 = 0.1$; (b) $\lambda_1 = 0.5$; (c) $\lambda_1 = 5$; (d) $\lambda_1 = 10$; (e) $\lambda_1 = 25$; (f) $\lambda_1 = 50$.

Now we consider a field of axisymmetric linear magnetic induction for which $\Psi_0(r,\theta)$ is

$$\Psi_0(r,\theta,\phi) = \frac{H_{11}}{3} r^3 \sin^2\theta\cos\theta \tag{4.136}$$

This corresponds to the case when $n = 2$ and $A_2 = \frac{H_{11}}{3}$ in the solution (4.119). If the magnetized sphere is introduced into this linear magnetic induction field the modified potentials can be constructed using the theorem 4.1 with $n = 2$. In the exterior phase we obtain

$$\Psi^e(r,\theta) = \frac{H_{11}}{3}\left[r^3 + \left(\frac{5f_2(\lambda a)}{\lambda a f_1(\lambda a)} - 1\right)\frac{a^5}{r^2}\right]\sin^2\theta\cos\theta \tag{4.137}$$

and in the interior phases the potential becomes

$$\Psi^i(r,\theta) = \frac{H_{11}}{3}\frac{5a^2}{\lambda a f_1(\lambda a)} f_2(\lambda r)\sin^2\theta\cos\theta \tag{4.138}$$

where $f_2(\lambda r)$ and $f_1(\lambda r)$ are the spherical Bessel functions of the first kind (see after (4.123)) and P_2^0, P_2^2 are associated Legendre polynomials.

We note that the image system in the exterior phase has a magnetic quadrupole located at the center of the sphere. The strength of the quadruple is given by $\left[\frac{5f_2)\lambda a)}{\lambda a f_1(\lambda a)} - 1\right]a^3$ and it depends on the penetration depth λ and the radius a.

The magnetic potential field for a linear induction is shown in Figure 4.12 for different values of λ_1 in xy-plane. In general, the pattern is symmetrical in the xy-plane. For small values of λ_1, the field lines are more inside the spherical conductor as seen in Figure 4.12(a)-(b). As λ_1 increases, the lines inside the sphere start getting close to the boundary as shown in Figure 4.12(c)-(d). For higher λ_1, the potential contours begin to stop penetrating the superconductors (see Figure 4.12(e)-(f)).

Indeed, when the penetration depth parameter λ is small, (or λ_1 is large), the field lines go around the sphere, indicating the superconductor limit as seen in Figure 4.12(f). In all cases, the field lines come from infinity are pushed back to infinity after interacting with the spherical boundary.

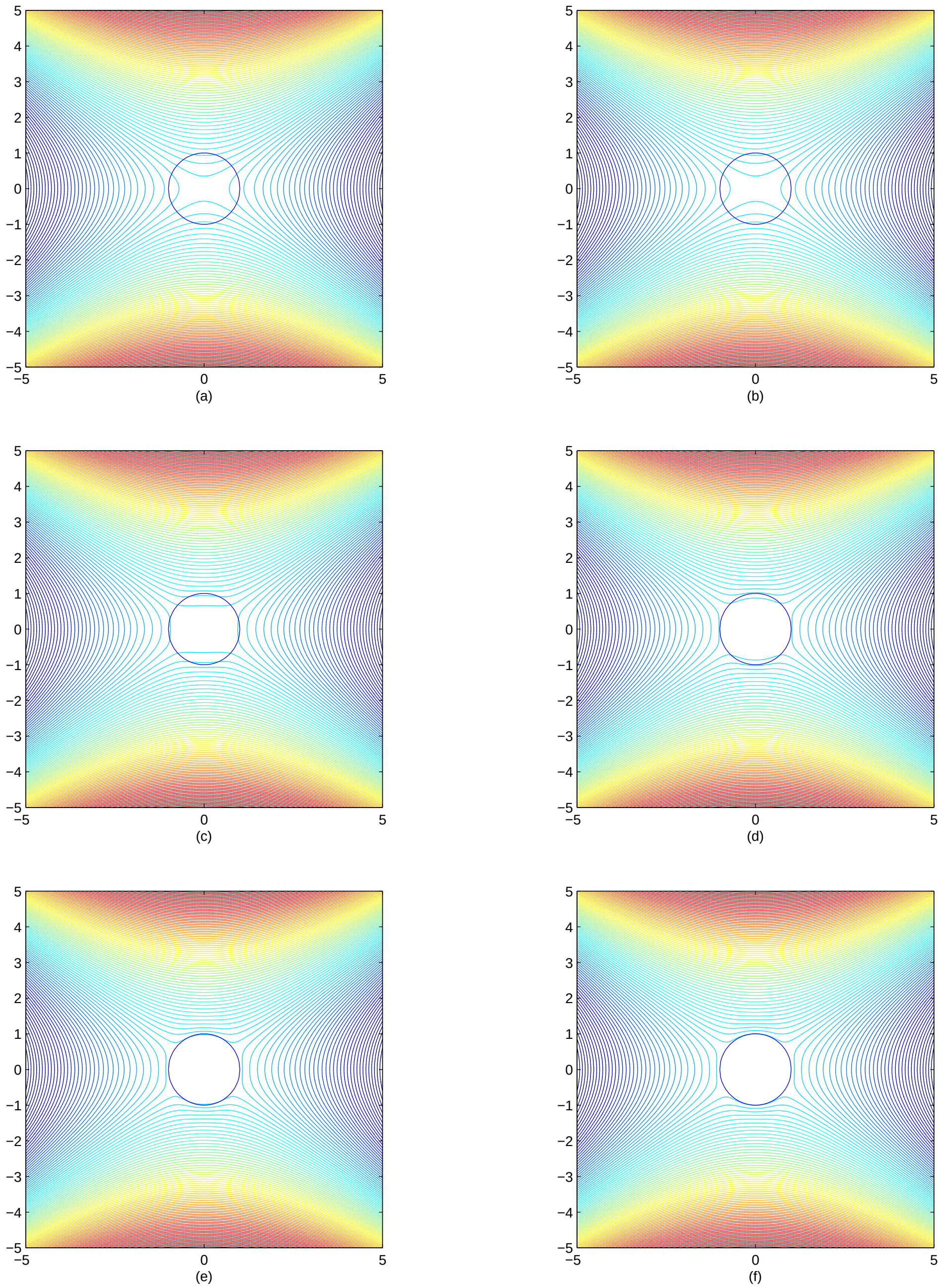

Figure 4.12
Potential plots for a sphere in a linear magnetic field using Maxwell-London model for various penetration depth (a) $\lambda_1 = 0.1$; (b) $\lambda_1 = 0.5$; (c) $\lambda_1 = 5$; (d) $\lambda_1 = 10$; (e) $\lambda_1 = 25$; (f) $\lambda_1 = 50$.

Significant changes occur for various values of the penetration depth λ. As λ varies from 0 to 1, the field lines move away from the center of the spherical superconductor. There are four sets of field contours occurring one in each quadrant. When λ increases from 0, the quadrupole pushes the contours away from the center towards the boundary of the superconductor. When λ is very large, the term with Bessel functions in (4.137) and (4.138) become vanishingly small and the potential function in the exterior phase reduces to that of the ideal superconducting sphere placed in a linear magnetic field in the Meissner state. This limiting case can also be observed in the contour plots given in figure $4.12(f)$.

CHAPTER V: SUMMARY AND CONCLUSIONS

A systematic approach for Maxwell-Maxwell and the Maxwell-London models is presented in this thesis. The potential functions in a two phase media with a spherical boundary in magnetostatics context are determined for MM and ML models and the general results are presented as theorems. The potential functions Φ^e and Φ^i in the exterior and interior phases, respectively, satisfy Laplace equations for the MM model. The axisymmetric potentials for the ML model satisfy the axisymmetric Laplace and Helmholtz equations in the respective phases. Both models lead mixed boundary value problems with Dirichlet and Neumann boundary conditions at the spherical surface $r = a$.

For the MM model five illustrative examples are discussed based on our general solutions. Namely, a magnetized sphere in (i) a constant magnetic field, (ii) linear magnetic induction field, (iii) a monopole field, (iv) a radial dipole field, and (v) a transverse dipole field. The image systems in each case is discussed in detail. The contour plots for the constant and linear fields show that the permeability parameter k has a significant effect on the magnetic fields. The plots show unphysical situation when $k > 1$ for which the interior magnetic permeability μ^i is negative. The force acting on the magnetized sphere is found for the monopole and dipole fields. It is shown that the force is positive for $k > \frac{1}{2}$ and negative for $k < \frac{1}{2}$. These results may be of interest in the design of levitating systems [28]. An interesting new relation for the multipole coefficients is derived. This relation demonstrates that the multipole coefficients for the MM model are a convex combination of Dirichlet and Neumann multipole coefficients for a sphere. The Clausius-Massotti factor in dielectrics is also connected to our multipole coefficients.

The axisymmetric flux density functions are determined in the exterior and interior phases for the ML model. The general solutions for Ψ^e and Ψ^i are given in infinite series forms and a theorem for ML model is stated and proved for a superconducting sphere placed in an external field. Exact solutions for constant and linear fields are derived from our new general solutions. The contour plots show that the penetration depth parameter λ has a significant impact on the magnetic fields. For larger values of λ our results indicate the limiting case of an ideal superconductor in Meissner state [10, 7]. Our solutions for the ML model do not cover the case when the fields are not axisymmetric. Extension of our results to asymmetric fields appears to be an open problem.

REFERENCES

[1] M. Abromowitz and I. A. Stegun, Handbook of Mathematical functions with formulas, graphs and mathematical tables. New York, Dover, (1972)

[2] M. Y Agre, Multipole expansions in magnetostatics, *Physics Uspekhi*, **54**, 167 - 180, (2011)

[3] R. G. Barrera, G. A. Estevez and J. Giraldo, Vector spherical harmonics and their application to magnetostatics, *Eur J. Phys.*, **6**, 287-294, (1985)

[4] Ronald Blaak and Jean-Pierre Hansen, Dielectric response of a polar fluid trapped in a spherical nanocavity, *J. Chem. Phys.*, **124**, 144-714, (2006)

[5] B. Carrascal, G. A. Estevez, Peilian Lee and V. Lorenzo, Vector spherical harmonics and their application to classical electrodynamics, *Eur J. Phys.*, **12** , 183-191, (1991)

[6] M. C. N. Fiolhais, H. Essen, C. Providencia, and A. B. Nordmark, Magnetic field and current are zero inside ideal conductors, *Progress In Electromagnetics Research B*, **27**, 187-212, (2011)

[7] D. J. Griffiths, *Introduction to electrodynamics*, 3rd edn. Addison-Wesley (1999)

[8] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Prentice-Hall, Amsterdam, (1983)

[9] T. Honegger, K. Berton, E. Picard, and D. Peyrade, Determination of Clausius-Mossotti Factors and Surface Capacitances for Colloidal Particles, *Appl. Phys. Lett.*, **98**, 181906, (2011)

[10] J. D. Jackson, *Classical Electrodynamics*, 3rd edn. John Wiley (1999)

[11] Q. Lin, Theoretical development of image method for superconducting spheres and cylinders, *Phys. Rev.*, **74**, 024510, (2007)

[12] I. V. Lindell and E. A. Lehtola, Magnetostatic Image Theory for the Permeable Sphere, *IEEE Trans. Magn.*, **28**, 1930-1934, (1992)

[13] I. V. Lindell, E. A. Lehtola and K. I. Nikoskinen, Magnetostatic Image Theory for an Arbitrary Current Loop in Front of a Permeable Sphere, *IEEE Trans. Magn.*, **29**, 2202-2206, (1993)

[14] F. London and H. London, The Electromagnetic Equations of the Superconductor, *Proc. R. Soc. London*, Ser. **A 149**, 71-88, (1935)

[15] S. Y. Lu and S. Kim, The general solution and Faxen laws for the temperature fields in and outside an isolated ellipsoid, *J. Engg. Math.*, **21**, 179-200, (1987)

[16] M. Ma, Z. Gan, and Z. Xu, Ion Structure Neat a Core-Shell Dielectric Nanoparticle, *Physical Rev. Lett.*, **118**, 076102, (2017)

[17] E. A. Matute., On the Superconducting Sphere in an External Magnetic Field, *Amer. J. Phys.*, **67**, 786-788, (1999)

[18] J. C. A. Maxwell, *A Treatise on Electricity and Magnetism*, 2nd Edn. Vol. 1, Calendron Press, Oxford, (1881)

[19] T. Myint-U and L. Debnath, *Partial Differential Equations for Scientists and Engineers*, Fourth Edition, Boston, Birkhauser, (2007)

[20] B. Nasmith and N. Gauthier, Electric fields in steady-state superconductivity according to London theory, *American Journal of Physics* , **76**, Issue 9 , 872-877, (2008)

[21] W. T. Norris, Charge Images in a Dielectric Sphere, *IEE Proc. Sci. Meas. Technol.*, **142**, 142-150, 1995

[22] D. Palaniappan, Magnetic interaction force and a couple on a superconducting sphere in an arbitrary dipole field. *J. Supercond. Novel Magn.*, **22**, 471-477, (2009)

[23] R. Saunders, A. Zisserman and C. J. McCauley, The Calculation of Magnetostatic Fields from Axisymmetric Conductors, *J. Phys. D: Appl. Phys.*, **29**, 533-539, (1996)

[24] S. N. Sazonov, Superconducting sphere in an external magnetic field revisited, *European Journal of Physics Education*, **4**, 1-10, (2013)

[25] M. Tejedor, H. Rubio, L. Elbaile and R. Iglesias, External Fields Created by Uniformly Magnetized Ellipsoids and Spheroids, *IEEE Trans. Magn.*, **31**, 830-836, (1995)

[26] C. Trombley, and D. Palaniappan, Interaction of a Straight Line Current with a Superconducting Sphere: Theoretical Results, *J. Supercond. Nov. Magn.*, **29**, 89-100, (2016)

[27] J. C. R. Turner, Two-Phase Conductivity, *Chem. Engg. Sci.*, **31**, 487-492, (1976)

[28] D. B. van Hulsteyn, A. G. Petschek, E. R. Flynn and W. C. Overton, Superconductor Imaging Surface Magnetometry, *Rev. Sci. Instrum.*, **66**, 3777-3784, (1995)

[29] P. Weiss, Applications of Kelvin's Transformation in Electricity, Magnetism, and hydrodynamics, *London, Edinburgh, Dublin Philos. Mag. J. Sci.*, **38**, 200-214, (1947)

[30] Z. Xu, Y. Liang, and X. Xing, Mellin Transform and Image Charge Method for Dielectric Sphere in an Ekectrolyte, *SIAM J. Appl. Math.*, **73**, 1396-1415, (2013)

[31] M. Y. Agre, Multipole expansions in magnetostatics, *Physics Uspekhi*, **54** , 167 - 180, (2011)

[32] G. C. K. Yeh, J. Martinek, and G. S. S. Ludford., The Potentials Due to Certain Singularities in The Presence of a Fixed Sphere, *J. Soc. Indust.Apll. Math.*, **3**, 142-152, (1955)

[33] G. C. K. Yeh, J. Martinek, and G. S. S. Ludford., A General Sphere Theorem for Hydrodynamics, Heat, Magnetism, and Electrostatics, *Z. Angew Math. Mech.*, **36**, 111-116, (1956)

[34] A. Zisserman, R. Saunders and J. Caldwell, Analytic Solutions for Axisymmetric Magnetostatic Systems Involving Iron, *IEEE Trans. Magn.*, **23**, 3895-3902, (1987)

NOMENCLATURE

r, θ, ϕ	spherical coordinates
$\Phi^e(r, \theta, \phi)$	exterior potential function
$\Phi^i(r, \theta, \phi)$	interior potential function
$r > a$	exterior phase
$r < a$	interior phase
μ^e	magnetic permeability in the exterior phase
μ^i	magnetic permeability in the interior phase
$k = \frac{\mu^e}{\mu^e + \mu^i}$	magnetic permeability parameter
ε_1	dielectric constant in the exterior phase
ε_2	dielectric constant in the interior phase
k_1	thermal conductivity in the exterior phase
k_2	thermal conductivity in the interior phase
λ	penetration depth
$\mathbf{B}^e$	Magnetic induction field in the exterior phase
$\mathbf{B}^i$	Magnetic induction field in the interior phase
$\Phi_0(r, \theta, \phi)$	Given potential in the absence of any boundaries
A_n, B_n, C_n	constant coefficients for the series solution for the MM model
$r^n S_n(\theta, \phi)$	spherical harmonic of degree n
$(0, 0, 0)$	center of the sphere
$(0, 0, c)$	location of the initial pole or dipole examples in cartesian coordinates, $c > a$

a	radius of the sphere
m	strength of a monopole
M_1, M_2, M_3	strengths of dipole
k_n	generalized Clausius-Massotti factor
γ	integration variable
$\hat{\mathbf{e}}_z$	unit vector in z-direction
$\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \ \hat{\mathbf{e}}_\phi$	unit vectors in r, θ, ϕ directions
$P_n^m(\cos\theta)$	Legendre polynomials of the second kind
$f_n(\lambda r) = \sqrt{\frac{\pi}{2\lambda r}} I_{n+\frac{1}{2}}(\lambda r)$	spherical Bessel function of the first kind
$I_n(\lambda r)$	modified Bessel function of the first kind (finite at the origin)
$K_n(\lambda r)$	modified Bessel function of the first kind (finite at infinity)