UNDERGRADUATE STUDENTS' DIFFICULTIES AND CONVICTIONS WITH MATHEMATICAL INDUCTION

A Thesis

by

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This thesis meets the standards for scope and quality of
Texas A&M University-Corpus Christi and is hereby approved.

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ABSTRACT

The study reported in this thesis has the purpose to gain insight into students' difficulties and self-described difficulties with mathematical induction and their convictions in their produced proofs. Participants were 78 undergraduate students from a four-year university. Three written tasks were administered, and interviews were conducted with eight of the participants. Two of the tasks asked the participants to prove two identities by mathematical induction and probed their conviction in their proofs. The third task asked the participants to describe their difficulties with proving said identities using mathematical induction. During the interviews, the participants were given an identity to prove using mathematical induction. Data analysis showed that students experienced difficulties at every step of mathematical induction, but mainly with stating $P(k + 1)$, algebraic manipulations needed to prove $P(k) \rightarrow P(k + 1)$, and writing the proof in the correct form. Students' self-described difficulties focused on the algebraic manipulations to prove $P(k) \rightarrow P(k + 1)$. With respect to their convictions in their proofs, students often derived conviction from one source – the proof or empirical evidence. Shifting the source of conviction from empirical evidence to the proof was uncommon.

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Formal proofs are used to construct and communicate mathematical knowledge within the community of mathematicians. The value of mathematical proofs is to justify mathematical statements; being convinced that a mathematical statement is true may help a student understand the proof. Undergraduate students first encounter formal proofs in introductory proof courses, like Discrete Mathematics. Mathematical induction is a method of proof that students learn in introductory proof courses. Proof by mathematical induction may be used to prove the truth of a statement, $P(n)$, that depends on a variable, n, where n is a natural number. Proofs by mathematical induction are used in advanced mathematics courses like Combinatorics and Modern Algebra. Mathematical induction is also important in Computer Science courses where sequences may be defined recursively or in closed forms.

The Purpose of the Study

Ernest (1984) provided an analysis of the Principle of Mathematical Induction (PMI) as well as the method of mathematical induction for use in the research of mathematics education. He stated that the correct usage of the method of mathematical induction includes the basis step, the inductive step, and the invocation of PMI. The construction of a proof by mathematical induction depends on the presentation of the proof in the correct form using the steps mentioned previously. Students may encounter difficulties at each step of a proof by mathematical induction.

Several studies have revealed some difficulties undergraduate students have with proofs by mathematical induction (Avital and Libeskind, 1978; Baker, 1996; Kong, 2003; Stylianides, Stylianides, and Philippou, 2007; Harel, 2001; Brown, 2003; Baker, 1995; Walter, 1972).

Weber and Mejia-Ramos (2015) stated that students may exhibit two types of conviction regarding his or her conviction of a mathematical proof. They defined absolute conviction as a student's complete certainty about the claim while relative conviction is defined by the subjectivity a student places upon the claim to determine certainty. For example, students who displayed relative conviction towards a claim may find an empirical argument convincing.

Given the importance of proofs by mathematical induction, the persistence of students' difficulties, and the different types of convictions students display with mathematical arguments, more research is needed regarding the teaching and learning of mathematical induction and students' convictions with mathematical induction as a method of proof.

The purpose of this study is to describe undergraduate students' difficulties with proof by mathematical induction and their conviction in mathematical induction as a method of proof.

Research Questions

This study answers the following research questions:

1. What are the difficulties that undergraduate students have when using the method of mathematical induction, as described by Ernest's (1984) framework?
2. How do students' self-described difficulties relate to how researchers describe students' difficulties with the method of mathematical induction?
3. What are students' convictions in mathematical induction as a method of proof?

Theoretical Framework

Adaptation of Ernest's Framework.

The E-a framework used in this study is described in this section.

Proofs by mathematical induction.

The framework provided by Ernest (1984) laying out the steps of mathematical induction for algebraic tasks has been used for this study to analyze students' responses to tasks:

1. Basis step – checking that $P(n_0)$ is true (n_0 is the initial value for which $P(n)$ is true);
2. Inductive step – checking that $P(k) \rightarrow P(k + 1)$ for some k, $k \geq n_0$;
3. Invocation of the Principle of Mathematical Induction (PMI) – based on the successful verification of the steps 1 and 2, we conclude that $P(n)$ is true for all n, $n \geq n_0$, by PMI.

Proving the basis step of induction relies on the ability to use the mathematical properties relating to $P(n_0)$ and performing algebraic substitutions. The ability to prove the induction step depends on stating the inductive hypothesis, $P(k)$, stating $P(k + 1)$, and using $P(k)$ to prove $P(k + 1)$ – a student may use the variable n in the inductive step, however it is common to introduce k to avoid difficulties with the circular logic of assuming what is to be proven. Presenting a proof by mathematical induction relies on verifying the first two steps and invoking PMI to conclude that $P(n)$ is true for all n, $n \geq n_0$.

To illustrate this framework, consider the task similar to a problem in Rosen (2012, p. 321):

$P(n)$: For all $n \geq 1$, $n \in \mathbb{Z}$, $n^3 - n$ is divisible by 3.

An example of a student's response using Ernest's (1984) framework may be the following:

1. Basis step:

   $P(1)$: $1^3 - 1 = 0$, and 0 is divisible by 3 since $0 = 3 \times 0$.

2. Inductive step:

   State $P(k)$ as "$P(k)$: $k^3 - k$ is divisible by 3 where k is greater than or equal to 1."

Assume $P(k)$ is true.

State $P(k + 1)$ as "$P(k + 1)$: $(k + 1)^3 - (k + 1)$ is divisible by 3."

We wish to show $P(k) \rightarrow P(k + 1)$.

Now use $P(k + 1)$ and simplify the expression $(\mathrm{k} + 1)^3 - (\mathrm{k} + 1)$:

$(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 - k + 3k^2 + 3k = k^3 - k + 3(k^2 + k)$

Since $P(k)$ is true, we can write $k^3 - k = 3m$ for some $m \in \mathbb{Z}$.

$k^3 - k + 3(k^2 + k) = 3m + 3(k^2 + k) = 3(m + k^2 + k) = 3j$, where $j = m + k^2 + k$ is an integer (since m and k are integers and $\mathbb{Z}$ is closed to addition and multiplication).

Therefore, $k^3 - k + 3(k^2 + k)$ is divisible by 3.

We have shown that $P(k)$ implies $P(k + 1)$.

3. Invocation of PMI:

Therefore, by PMI, the statement, $n^3 - n$ is divisible by 3 for all $n \geq 1$, is true.

Difficulties with constructing a proof by mathematical induction.

Ernest (1984) stated that there are certain abilities required by students to construct a proof by mathematical induction. Difficulties with these abilities are in line with the steps he laid out for a successful proof by mathematical induction. Figure 1 shows how these difficulties occur at a step in a proof by mathematical induction.

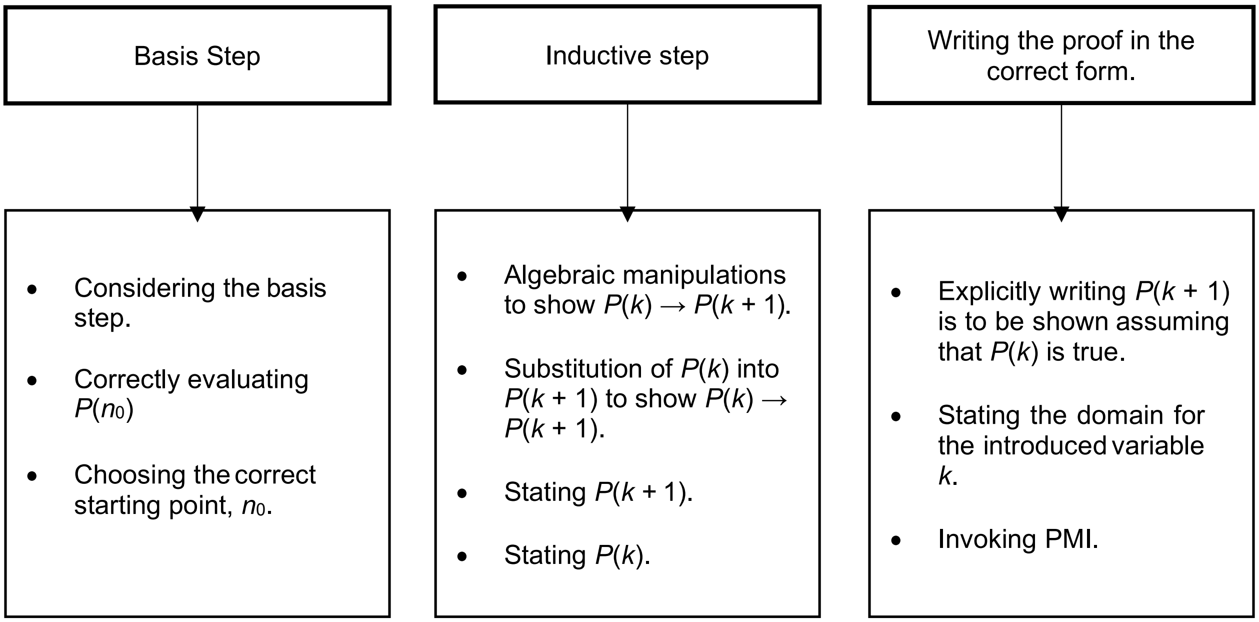


Figure 1. Student difficulties with constructing a proof with by mathematical induction.

Figure 1 shows how a student can encounter a difficulty at a step, or sub-step, during the construction of a proof by mathematical induction. The arrows relate the sub-steps belonging to the main step, the basis step, inductive step, or writing the proof in the correct form. To see examples of these difficulties, consider the task:

Prove by induction the statement $P(n)$: $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, where $n \in \mathbb{N}$, $n \geq 1$ (Rosen, 2012, p. 317).

- Difficulties with the basis step.
    - Omitting the basis step. For example, a student's proof may look like:

      $P(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$, $k \in \mathbb{N}$, $k \geq 1$.

      Assume $P(k)$ is true.

      $P(k + 1)$: $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

      We wish to show $P(k) \rightarrow P(k + 1)$.

Using $P(k + 1)$:

$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$.

$k^2 + 2k + 1 = (k + 1)^2$.

$(k + 1)^2 = (k + 1)^2$.

Therefore, by PMI, $P(n)$ is true.

- Choosing the incorrect n_0. For example, a student may choose 0 instead of 1 which will result in:

  $P(0)$: $-1 = (-1)^2$.

  However, $-1 \neq 1$.

- Incorrectly evaluating the basis step. For example, a student may substitute 1 into $P(n)$ but evaluate it incorrectly:

  $P(1)$: $2(1) - 1 = 1^2$.

  However, $2 \neq 1$.

  The student might then discontinue the proof or claim the statement, $P(n)$, to be false because $P(1)$ was considered false.

- Difficulties with proving the implication $P(k) \rightarrow P(k + 1)$.
  - Difficulty stating $P(k)$. For example, a student's answer could look like:

    $P(k)$: $(2k - 1) = k^2$.

    Here, the student concerns himself or herself with the last term in the sum.
  - Difficulty stating $P(k + 1)$. For example, a student attempting to state $P(k + 1)$ may write:

    $P(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$, $k \in \mathbb{N}$, $k \geq 1$.

    Assume $P(k)$ is true.

$P(k + 1)$: $1 + 3 + 5 + \ldots + (2k) = k^2 + 1$.

- Difficulty manipulating algebraically the terms in $P(k)$ and $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$. For example, in stating $P(k + 1)$, a student will have the term $(k + 1)^2$ to possibly expand or the terms $k^2 + 2k + 1$ to possibly factor in showing that $P(k) \rightarrow P(k + 1)$:

  $P(1)$: $1 = 1^2$. True.

  $P(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$, $k \in \mathbb{N}$, $k \geq 1$.

  Assume $P(k)$ is true.

  $P(k + 1)$: $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

  We wish to show $P(k) \rightarrow P(k + 1)$.

  Using $P(k + 1)$:

  $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$.

  Since $P(k)$ is assumed to be true, we can write:

  $k^2 + (2k + 1) = (k + 1)^2$.

  $k^2 + 2k + 1 = (k + 1)^2$.

  $k^2 + 2k + 1 = k^2 + 1^2$.

- Difficulty substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$. For example, a student might write:

  $P(1)$: $1 = 1^2$. True.

  $P(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$, $k \in \mathbb{N}$, $k \geq 1$.

  Assume $P(k)$ is true.

  $P(k + 1)$: $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

  We wish to show $P(k) \rightarrow P(k + 1)$.

Using $P(k + 1)$:

$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$.

$k^2 + (2k + 1) = (k + 1)^2$.

$(k + 1)^2 = (k + 1)^2$.

- Difficulties with writing the proof in the correct form.
  - Not invoking PMI. For example, a student may write:

    $P(1)$: $1 = 1^2$. True.

    $P(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$, $k \in \mathbb{N}$, $k \geq 1$.

    Assume $P(k)$ is true.

    $P(k + 1)$: $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

    We wish to show $P(k) \rightarrow P(k + 1)$.

    Using $P(k + 1)$:

    $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$.

    $k^2 + 2k + 1 = (k + 1)^2$.

    $(k + 1)^2 = (k + 1)^2$.

    Therefore, $P(n)$ is true.

  - Not explicitly writing that $P(k + 1)$ is to be shown assuming that $P(k)$ is true. For example, a student may write:

    $P(1)$: $1 = 1^2$. True.

    $P(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$, $k \in \mathbb{N}$, $k \geq 1$.

    $P(k + 1)$: $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

    Using $P(k + 1)$:

    $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$.

$k^2 + 2k + 1 = (k + 1)^2$.

$(k + 1)^2 = (k + 1)^2$.

Therefore, $P(n)$ is true by PMI.

- Not stating the domain for the introduced variable k. For example, a student might write:

  $P(1)$: $1 = 1^2$. True.

  $P(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$.

  Assume $P(k)$ is true.

  $P(k + 1)$: $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

  We wish to show $P(k) \rightarrow P(k + 1)$.

  Using $P(k + 1)$:

  $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$.

  $k^2 + 2k + 1 = (k + 1)^2$.

  $(k + 1)^2 = (k + 1)^2$.

  Therefore, $P(n)$ is true, by PMI.

This adapted framework of Ernest (1984) has been called the E-a framework.

Weber and Mejia-Ramos' Enriched Framework.

Consider, again, the task similar to a problem in Rosen (2012, p. 321):

$P(n)$: For all $n \geq 1$, $n \in \mathbb{Z}$, $n^3 - n$ is divisible by 3.

To begin, Weber and Mejia-Ramos (2015) stated that students may exhibit two types of conviction regarding his or her conviction of a mathematical proof. Weber and Mejia-Ramos (2015) defined their categories as:

- Absolute conviction – a student is completely convinced about a claim and has no doubt about its validity. For example, a student, who was presented with a proof that $P(n)$ is true for any integer, n, greater than or equal to 1, when asked whether the statement was true for $n = 4$ would respond by saying:

  Yes, the statement is true for $n = 4$ because the presented proof has proven it.

- Relative conviction – a student is not completely convinced and requires additional actions until conviction in the statement is reached. For example, a student presented with a proof that $P(n)$ is true for any n integer greater than or equal to 1, when asked whether the statement was true for $n = 4$, would check that $P(4)$ is true.

  $P(4)$: $4^3 - 4$ is divisible by 3.

  Since $4^3 - 4 = 60$ is divisible by 3, the statement is true for $n = 4$.

If a student's response is not within one of the categories from above, there exists no category into which the response may be placed. For example, consider a student's response when asked whether $P(0)$ is true:

$P(0)$ is not true because the proof only proves $n^3 - n$ is divisible by 3 for all integers greater than or equal to 1 and 0 is not a positive integer.

However, $0^3 - 0 = 0$ which is divisible by 3, so $P(0)$ is indeed true, although the proof for $P(n)$ is true refers only to integers greater than 1. Now, consider the following example of a student's response when asked whether the statement, $P(n)$, is true for $n = 4$:

$P(4)$: $4^3 - 4$ is not divisible by 3.

$4^3 - 4 = 64$ is not divisible by 3 since $63 = 3 \times 21$ and $66 = 3 \times 22$, and there is no integer between 21 and 22, therefore $P(4)$ is not true.

In this example, an error was made when checking the validity of $P(4)$. It was determined that $P(4)$ was false despite having been presented with a proof that $P(n)$ was true for all integers greater than or equal to 1. Due to these responses, additional categories are required to accurately describe students' convictions towards proofs by mathematical induction.

To analyze students' convictions with proofs by induction, an enriched framework based on Weber and Mejia-Ramos (2015) has been used for this study. This has been called the Weber and Mejia-Ramos enriched framework (WMRe). Figure 2 shows this enrichment.

Weber and Mejia-Ramos (2015) WMRe

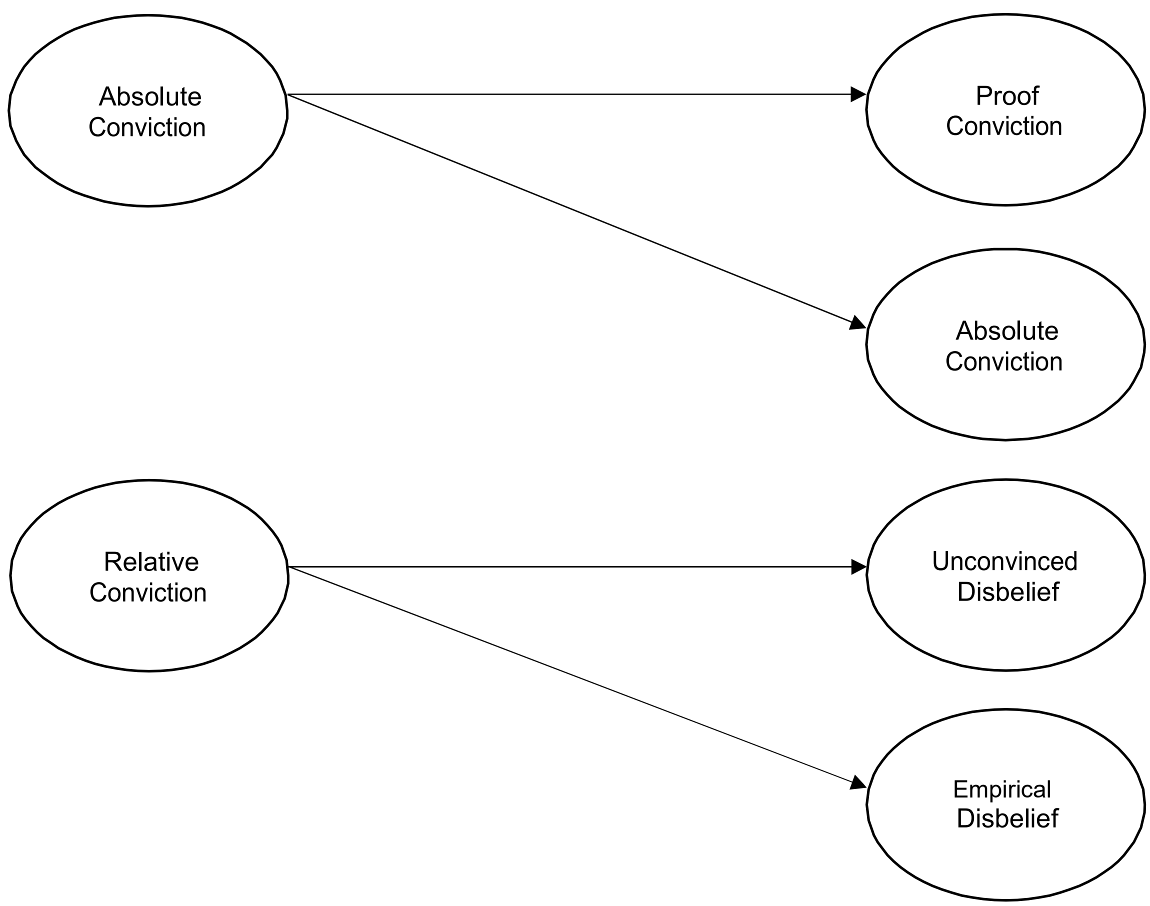


Figure 2. The enrichment of Weber and Mejia-Ramos (2015).

Figure 2 displays how each category described from Weber and Mejia-Ramos (2015) can be split into two categories for a total of four categories. For this study, the term “conviction” defines a student’s certainty regarding the statement and his or her proof. Should a student have no conviction or a lack of conviction in his or her proof, the instance has been defined as “disbelief.” To provide examples for how students’ responses may be categorized using the WMRe framework, recall the task similar to Rosen (2012, p. 321):

$P(n)$: For all $n \geq 1$, $n \in \mathbb{Z}$, $n^3 - n$ is divisible by 3.

The following are the categories in the WMRe framework that have been used in this study, accompanied by examples of student responses from the task previously mentioned:

- Proof Conviction – a refinement of “absolute conviction” defined by Weber and Mejia-Ramos (2015) by identifying the source of conviction. A student derives his or her conviction from his or her proof of the understood statement. The student understands what his or her proof has shown. For example, a student asked whether the statement, $P(n)$, was true for $n = 4$ may respond by:

  > My proof has shown $P(n)$ is true for all integers greater than or equal to 1 and 4 is an integer greater than 1, therefore the statement $P(4)$ is true.

- Absolute Conviction – a student derives his or her conviction from the proof of the understood statement. However, he or she is convinced that for any value not within the proven domain associated with the statement, the statement is false. For example, a student asked whether the statement was true for $n = 0$ would respond by saying:

  > The proof was only done for integers greater than or equal 1, and 0 is not greater than 1, therefore, $P(0)$ is not true.

- Unconvinced Disbelief – a refinement of “relative conviction” defined by Weber and Mejia-Ramos (2015) by identifying the lack of conviction, correctness of conclusion, and method of conclusion. The proof produced by the student does not aid in establishing conviction and justification is provided that does not use the proof. However, a student may feel the need to provide an additional verification that is different from his or her proof after having already proved the statement. More information from the student regarding the tasks would be needed to justify placing the student’s response in this category. For example, a student asked whether the statement, $P(n)$, was true for $n = 4$ would respond by saying:

  $P(4)$: $4^3 – 4$ is divisible by 3.

  $4^3 – 4 = 60$. Since $60 = 3 \times 20$, therefore $P(4)$ is true.

  To justify categorizing the student’s response here, an interview with the student would be needed or an additional task asking the student about the difficulties with or comments about the task asking whether the statement was true for $n = 4$ to determine whether the student only used the proof as a formality. For example, a student’s response that truly displayed Unconvinced Disbelief would respond to a question asking about the difficulties with determining or commenting on whether $P(4)$ is a true statement by similarly stating:

  - The statement, $P(4)$, is true because I got 60 which is divisible 3.

  The students’ responses are placed in this category based on this, or similar, response because it is demonstrated that conviction is not derived from the produced proof, but rather by criteria unrelated to the proof such as empirical evidence. A student’s response in this category would need to satisfy three criteria:

1. Prove the statement, $P(n)$, by mathematical induction.
2. Determine that the statement, $P(a)$, is true where $a \in \mathbb{Z}$ such that $a \geq n_0$. This may be done by using empirical evidence or stating a reason that does not include the proof.
3. Provide additional evidence that conditions used to establish conviction did not include the proof. This may be done by obtaining evidence from the student about the task in the form of a written or verbal response to a probing question.

- Empirical Disbelief – a misunderstanding of definitions used by the statement, the statement itself, or is unconvinced by his or her proof. In doing so, conviction is derived from empirical calculations whose result is contradictory to the proof or the statement. For example, a student asked whether the statement was true for $n = 4$ would respond by saying:

  $P(4)$: $4^3 - 4$ is divisible by 3.

  However, since $63 = 3 \times 21$ and $66 = 3 \times 22$, and there is no integer between 21 and 22, therefore $P(4)$ is not true.

  For example, a student's response displaying Empirical Disbelief would respond to a question asking about the difficulties with determining or commenting on whether $P(4)$ is a true statement by stating:

  I feel that $P(4)$ is true, but I am getting that it is false.

  The student's response is placed in this category because it shown that the student is relying on the result of the calculation for conviction rather than the proof. A student's response placed in this category would need to satisfy:

1. Prove the statement, $P(n)$, by mathematical induction.
2. Determine that the statement, $P(a)$, is false where $a \in \mathbb{Z}$ such that $a \geq n_0$. This is commonly done by relying on the empirical calculations done by the student.

In the next chapter, the review of literature is presented by student difficulties with mathematical induction and then students' conviction, certainty, and beliefs with mathematical arguments.

Chapter 2: Review of the Literature

Many of the studies that focused on the teaching and learning of mathematical induction as well as student difficulties occurred in or before the 1990s (Avital and Hansen, 1976; Avital and Libeskind, 1978; Ernest, 1984; Reid, 1992; Walter, 1972; Dubinsky and Lewin, 1986; Segal, 1998; Kahn, Anderson, Austin, Barnard, Jagger, and Chetwynd, 1998; Anderson, Austin, Barnard, and Jagger, 1998). There is now a noticeable lack of research on student difficulties with proof by mathematical induction in the last decade with most studies only using one task as a means to research another topic. These studies have focused on new teaching methods (Allen, 2001; Harel, 2001; Powers, Allison, and Grassl, 2005; McAndrew, 2010; Webber, 2012; Brown, 2014; Wang, Wang, Li, and Rugh, 2018) or students' abilities to construct or understand arguments (Pedemonte and Buchbinder, 2011; Tabach, Barkai, Tsamir, Tirosh, Dreyfus, and Levenson, 2010; Martinez and Pedemonte, 2014; Garcia-Martinez and Parraguez, 2017).

Studies Regarding Students' Difficulties with Mathematical Induction

Baker (1996) conducted a study with 13 undergraduate students in Indiana. The college students' mathematical backgrounds varied from high school mathematics to beyond college calculus and were recruited from a computer science course in digital computing. The researcher investigated the characteristics of students' difficulties with mathematical induction. Data were collected through a questionnaire. The questionnaire had a proof writing task, four proof analysis tasks, and three general questions about proof and mathematical induction. The researcher found that inadequate prior mathematical knowledge needed to answer the questionnaire caused many problems for students. The tasks required students to be able to expand the summation symbol, expand the factorial symbol, know the definition of divisibility, understand the concept of a variable, and know how to simplify algebraic expressions using the exponent rules. Deductive logic caused problems for all participants. There were three ways deductive logic caused problems: reliance on informal rules of logic, connection between basis step and validity of the induction hypothesis, and the inductive step applying to all cases of the statement. For example, half of the 13 undergraduate students did not recognize a missing basis step and over a third of them did not understand that the inductive step must be generalized for all cases (i.e. no empiricism or specific cases). Baker (1996) found that students relied on procedures and failed to understand what was proven. The students also used examples to convince themselves of how to go about the proof and verify statements. Baker (1996) claimed that students focus on procedure rather than how each step is used for mathematical induction to create a mathematical proof and the primary source of difficulty was prior mathematical knowledge required to complete the inductive step of mathematical induction. He also found that students who completed more mathematical courses scored higher on the tasks.

According to Baker (1996), the participants were only given two weeks of instruction on mathematical induction. This is not an adequate amount of time to learn a method of proof considered by researchers to be difficult. With only 13 undergraduate students, generalizations cannot be made about their difficulties with mathematical induction or with their understanding of mathematical induction.

Harel (2001) conducted a teaching experiment to understand the development of students' proof schemes regarding mathematical induction. The teaching experiment consisted of six sessions with 25 prospective secondary teachers. Data were collected through video-taped classroom sessions, homework, interviews, exams, quizzes, and field notes. Harel (2001) stated that mathematical induction is traditionally introduced using generalizations of a single variable, and then the proof of the sum of the first *n* natural numbers is shown as an example. The basis step and inductive step are then laid out along with the common misconceptions to help students avoid errors. Standard textbooks divide mathematical induction problems into implicit recursion problems, explicit recursion problems, and non-recursion problems. Harel (2001) hypothesized that introducing students to different types of problems in a certain order promoted the development of inductive reasoning and give meaning to mathematical induction as a method of proof. The order of problems given to students during the teaching experiment was implicit recursion problems, then explicit recursion problems, and finally general problems requiring mathematical induction such as divisibility and inequalities. He believed that throughout this process of inductive reasoning with these problems, students developed the intellectual need to formulate and communicate ideas - the need for convincing themselves and others about their observations. Harel (2001), therefore, recommended that when teaching mathematical induction to students, one should begin with recursive problems which are generally put towards the end of the chapter in textbooks.

One limitation from his experiment is that students did not completely develop the method or Principle of Mathematical Induction themselves, instead the students were explicitly given the Principle of Mathematical Induction. Also, the author did not comment on the difficulties the students had with mathematical induction, but rather commented on the development of inductive reasoning through generalizations. This teaching experiment also put forth a teaching method emphasizing recursion, so further research is required to verify these developments and difficulties on undergraduate students.

Brown (2003) conducted a teaching experiment with six Calculus II students. These six students were invited to participate on the basis of good algebra skills and no prior knowledge of mathematical induction. Brown's (2003) goal for the study was to use the teaching experiment to determine which students' difficulties described in literature arose when using proof by mathematical induction, and the difficulties students have when learning proof by mathematical induction. The author identified that other researchers say that PMI is difficult to understand for students, but mathematicians have referred to it as intuitive. Brown (2003) claimed that proofs are explanatory, but students consider alternative methods as "an explanation." An example of this is a student's view on a geometric representation as a formal solution, an explanation, to the sum of the first n integers. A difficulty found was the students' inability to find a "representation of a successor" with sequences and series. Another problem the author found was that students considered the problem solved when they found a closed form expression, and when they used specific cases with inductive reasoning to justify their proof. Students were given the Chords of a Circle task:

> "Suppose you have a circle with n points marked on the circumference. By connecting each pair of points with straight-line segments the circle can be partitioned into [regions]. Is there a function for calculating the number of regions?"

When the students encountered a problem that did not work for all tested cases, like in the Chords of a Circle task, the students were convinced it was a trick question. Brown (2003) stated that the reactions of the students are evidence that students rely on the teacher in such a way that teacher-posed questions of existence imply existence. The students were also found to show difficulty in understanding the implication statement of problems they encountered. Students were given the L-tiling task:

> "For any positive integer n, can a $2^n \times 2^n$ grid with one square removed be tiled with L-tiles?"

In proving the L-tiling task, the students did not articulate that “the one before could be L-tiled.” Brown (2003) claimed that this difficulty with assumptions, or inductive hypotheses, occurs because mathematics is thought of as simply a collection of truths. The students did not see mathematics as having hypotheses.

The teaching experiment was conducted with only six students, so more research needs to be done to confirm the findings and analyses Brown (2003) provided. Since the study was about the understanding of the development of mathematical induction, more emphasis would need to be done on students who have had exposure to mathematical induction.

Stylianides, Stylianides, and Philippou (2007) conducted a study at the University of Cyprus with 95 preservice teachers consisting of 70 education majors and 25 mathematics majors. These students were preparing to become elementary school teachers, and secondary school teachers, respectively. A consequence of the study was discovering how preservice teachers understand mathematical induction and proofs in general. The instruments the researchers used consisted of a test, but only two tasks were used for analysis, and interviews with eight education majors and three mathematics majors. The first task, Task 1, showed students a proof by mathematical induction of the statement:

“For every $n \in \mathbb{N}$ the following is true: $1 + 3 + 5 + \ldots + (2n - 1) = n^2 + 3$.”

The proof was missing the basis step. The students needed to respond using the following four options to justify their choices: i) the proof is invalid; ii) the proof shows that the statement is always true; iii) the proof shows that the statement is true in some cases; iv) I have no opinion. For Task 1, Stylianides et al. (2007) found that nearly all mathematics majors answered correctly, but only a quarter of the education majors correctly identified the proof as invalid. The interviews demonstrated students’ difficulties and misunderstandings. For example, students’ interviews showed that although they could identify that the basis step was missing, they neither checked it nor could explain its necessity. Interviews also showed that the students believed that although the proof did not prove the statement for the specified domain of the variable n, a new domain could be found to make the proof valid. In other words, the students believed that the domain was faulty, not the proof. The second task, Task 2, presented a correct proof by mathematical induction of the statement:

"For every natural number $n \geq 5$, the follow is true: $1 \times 2 \times \ldots \times (n - 1) \times n > 2^n$"

and again, Task 2 asked students to comment on the validity of the proof using the four choices described above in Task 1. Next, students were asked to state whether the inequality was true for the values $n = 3$, $n = 4$, $n = 6$, and $n = 10$. The majority of mathematics majors correctly answered that "the proof showed that the statement was always true," while only half of the education majors answered correctly. The researchers inferred that the education majors may have leaned more towards the choice that the proof showed the statement was true in some cases because the domain of the statement in Task 2 did not include all possible values for which the statement, $P(n)$,was true. The interviews also showed that some students considered the statements $P(3)$ and $P(4)$ to be automatically false without verification because the values 3 and 4 of the variable n were not in domain of the proven statement, $P(n)$, in Task 2. Stylianides et al. (2007) hypothesized that these difficulties and misunderstandings come from prior experiences with didactic contracts where the basis step is always true, and proven statements with their domains, for n, always include every possible value of n for which the statement is true. They also claimed that their findings extend to university students of any discipline.

Further research needs to be done on undergraduate students to extend the knowledge of students' difficulties regarding mathematical induction. Stylianides et al. (2007) did not ask the students about their difficulties, only inferring their difficulties from the students' work.

Kong (2003) conducted a study at Tampines Junior College in Singapore with 30 first year students who were 17 years old. The purpose of the study was to determine the relationship between students' performance and the types of identities needing to be proven, and the difficulties students encounter when learning mathematical induction. The questionnaire contained three questions regarding mathematical induction on two sums involving the summation symbol, and the formulation and subsequent proof of a conjecture. The first question involving the summation symbol was:

"Show by induction that, for every positive integer n, $\sum_{r=1}^{n} \frac{r+1}{2^r} = 3 - \frac{1}{2^n} \times (n+3)$."

The second question involving the summation symbol was:

"Prove by induction that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)(r+5)} = \frac{11}{96} - \frac{1}{4(n+2)(n+4)} - \frac{1}{4(n+3)(n+5)}$."

Each question was scored from 0 to 10, where 10 was the highest possible score. The researcher found that students scored 5.70 and 5.67 on the two questions involving the summation symbol. Students were asked:

"Let (x_n) be a sequence of numbers defined by: $x_1 = 1$, $x_{n+1} = 2^n x_n$ for $n = 1, 2, \ldots$. Find a formula for and x_n prove it by mathematical induction."

It was found that they performed extremely poorly when they had to formulate a conjecture, with a mean 0.80. The question that required the formulation of a conjecture was the most difficult for students. The students' success in the inductive step relied on the ability to correctly assume $P(k)$ and use it to show that $P(k) \rightarrow P(k + 1)$. Another difficulty discovered was that the lack of mathematical content knowledge caused problems for students. Specifically, the content knowledge from the understanding of the definition of a summation and a sequence, relating to the three problems already mentioned. The author claimed that the inability to formulate a conjecture is evidence of students' difficulties with mathematical induction regarding content knowledge. Kong (2003) argued that students' prior mathematical knowledge is an important aspect in a successful proof by mathematical induction. He also believed that the implication $P(k) \rightarrow P(k + 1)$ is where students fail the most because of the lack of prior mathematical knowledge.

The students' inability to formulate a conjecture and then proving it by mathematical induction is a deficiency in inductive reasoning. It's an overgeneralization with inadequate research about students' abilities to problem solve.

Lane (2010) conducted a study with 78 students from three different universities. He studied how mathematical induction relates to implication functions and proposition functions. A written assignment was given to the students that had 14 questions on it, seven questions about functions and seven about mathematical induction. An example of a question on the written assignment related to mathematical induction was a proof by mathematical induction of the statement was:

"$10n \geq \frac{n + 10}{n + 1}$ for $n = 1, 2, 3, \ldots$"

where students were asked to explain several steps of the proof. He found that the students who did well on the ACT Math exam and questions related to functions performed well on the questions with mathematical induction. Lane (2010) concluded that, from his study, the correlation between mathematical induction and function ability was $r = 0.47$ ($p \leq .001$). He also found that the school a student attended, the major he or she pursued, and the mathematics course he or she was currently taking did not influence the student's ability with mathematical induction. From the interviews the researcher conducted, he found that students did not know when proof by mathematical induction could be used. The interviewed students gave explanations like "[when] any proposition that involves a systematic process where you can use the previous term to help prove the next…" and "any situation involving an infinite set" (Lane, 2010, p. 139). Lane (2010) recommended stressing conceptual understanding over procedural as many participants wrote all steps but could not explain proof by mathematical induction.

Anderson, Austin, Barnard, and Jagger (1998) administered a test to 155 third-year undergraduates from 15 different universities in England. The purpose of the study was to see the extent of retention of the content from first year university mathematics courses. One question on the test was:

> "Prove, by the method of mathematical induction, that for all positive integers n, $4^n + 6n - 1$ is a multiple of 9. Explain, in your own words, why your proof shows that the result is true."

The results showed that 19 students either left the question completely blank or responded that they had forgotten the method. Only 26 students received the highest score (an A) on the question which required them to give a satisfactory explanation of their proof. The students demonstrated a reliance on the procedure of mathematical induction and excluded an explanation. They also showed that they had trouble explaining the purpose of the in the inductive step – a common response being "prove it true for $n = 1$, assume it true for $n = k$, prove it true for $n = k + 1$" or something similar. The authors also found that students were sloppy with writing the steps of mathematical induction such as $P(n)$ meaning both the proposition and the expression $4^n + 6n - 1$ therefore writing something similar to "$P(1) = 4 + 6 - 1 = 9$, so $P(1)$ is true." With these results, Anderson et al. (1998) claimed that mathematics taught at the beginning in the first year of mathematics courses is neither understood nor retained. Theauthors suggested that students do not form connections proofs as the students did not attempt to reconstruct mathematical knowledge from courses already completed. They also claimed that students' motivation in doing well at mathematics stems from whether they can perfect techniques rather than understanding.

Demiray and Bostan (2017) studied teachers' abilities to conduct valid mathematical proofs and the proof methods they used. The study comprised of 115 Turkish preservice middle school mathematics teachers who were also undergraduate students. The participants were given two questions that may be proven using mathematical induction. The first question was:

"Show that $1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}$."

Proof by mathematical induction was the most frequent method of proof with 50 preservice teachers using it. Twelve participants produced invalid arguments because of incomplete proofs by mathematical induction. The second question was:

"For all natural numbers, n, $3|(n^3 - n)$."

Again, proof by mathematical induction was the most frequent proof method with 37 preservice teachers using it. As before, 12 participants produced incomplete proofs by mathematical induction. Demiray and Bostan (2017) state that the reason for the 24 incomplete proofs by mathematical induction was the result of the inability to use $P(k)$ to prove $P(k + 1)$. The authors claimed that this may be due to preservice middle school mathematics teachers' tendencies to memorize proof methods instead of understanding the logic of proofs.

Dubinsky and Lewin (1986) studied 22 mathematics majors understanding of mathematical induction. These students were recruited from an advanced mathematics course. Data were collected by individual interviews where students were asked to explain their proofs by mathematical induction. If students were able to provide an explanation, they were asked to apply it to a mathematical statement. The mathematical statement given to students was to prove by induction that:

$$x_{n+1} = \sqrt{1 + x_n} \text{ for all } n \geq 1 \text{ where } x_1 = \sqrt{2}.$$

The researchers found that students would attempt to explain something and look at the interviewer for approval or continue explaining until the interviewer approved. From the interviews, they saw that students could explain verifying the basis step but could not explain the implication statement for the inductive step or its importance. Students were also found to be able to explain the idea of mathematical induction but could not apply it to the problem. Dubinsky and Lewin (1986) also found that it's possible for a student to understand and explain mathematical induction, but he or she won't use it as a method of proof. The authors explained these difficulties as students' inabilities to assimilate all components of proof by mathematical induction into their proof schemes.

Stylianides, Sandefur, and Watson (2016) conducted a study with 12 undergraduate students from a private American university. Four trios were created from the 12 students. The authors studied students' abilities to develop a conjecture with proof. Data collected comprised of video and audio recorded interviews with the students and each trio's work on a problem. The first problem given to two of the trios was:

> "Develop a conjecture about the remainder when 5^n is divided by 3 and prove your conjecture."

One trio quickly made a table of values to make a correct conjecture, and one student identified induction as a possible proof method. They were able to split the proof by mathematical induction into even and odd cases for the exponent. Another trio also attempted use mathematical induction. However, the trio was unable to complete their proof because they did not develop the correct induction hypothesis; they also made several algebraic mistakes. Both trios were found to have difficulty with implementing division by 3. The second problem was given to two different trios and asked:

> "For what odd values of n is the finite geometric series, $\sum_{i=0}^{n} 2^i$ a prime number? Prove your conjecture."

One trio made a table of values and saw that each one was divisible by 3. The students in the trio exhibited confusion about using only odd numbers and difficulty proceeding with mathematical induction because what they knew, each value was divisible by 3, did not match what they were trying to prove, having only one value that was prime. The other trio also made a table of values but knew that the sum was equal to $2^{n+1} - 1$ and wrote the sums with the general formula. They were unconvinced that using mathematical induction could prove the conjecture and had to prompted by the interviewer to use it. The authors stated that the last two trios used examples to verify and expose structural relationships. Stylianides et al. (2016) claimed that the students did not have difficulty in using the method of mathematical induction but, rather, in the manipulation and formulation expressions related to the statements. They stated that prior knowledge of number theory accounted for students' weaknesses and strengths with respect to the problems. They also claimed that recursive problems allow students to see mathematical induction as explanatory because students are required to formulate iterations.

In short, a review of studies on students' difficulties shows that difficulties can occur at any step of a proof by mathematical induction. Students have trouble understanding the necessity of the basis step (Avital and Libeskind, 1978; Lane, 2010), forget to verify the basis step (Stylianides, Stylianides, and Philippou, 2007), do not understand the implication statement (Ernest, 1984; Baker, 1996; Brown, 2003; Lane, 2010), and lack prior mathematical knowledge necessary in the inductive step (Baker, 1996; Kong, 2003). The research discussed above has neglected to question students about their perceived difficulties with mathematical induction after having had instruction on the topic. Nevertheless, should students produce proofs by mathematical induction, the next questions may be whether they understand what was concluded or, more interestingly, their conviction in it.

Studies Regarding a Student's Conviction, Certainty, or Belief Towards Mathematics

A student who proves a mathematical statement should be convinced by the conclusion of his or her proof. However, students have been known to adopt a ritualistic proof scheme (Harel and Sowder, 1998) to appease educators – simply following the steps that will produce the result desired by the requester. Inculcation of the steps of mathematical induction does not provide conviction for students; it only provides a checklist for students to complete to ensure correctness. It falls on students to fill this gap between showing what someone requests and believing what they have just concluded. Oftentimes, students use personal reasons for conviction while following mathematical standards and authority for the validity of arguments (Knuth, 2002).

Knuth (2002) conducted a study with 16 in-service secondary school mathematics teachers. Eleven participants had undergraduate mathematics degrees while five had undergraduate engineering or physical science degrees. Data were collected through semi-structured interviews. The participants were asked "Do proofs ever become invalid?" Four of the 16 teachers stated that a proof is not subject to contradictory evidence. Six other teachers said that the proof may become invalid if the axiomatic system under which the proof was constructed is changed. Nevertheless, all teachers suggested that mathematical proofs establish the truth of a statement with 11 qualifying it by means of logic or deductive reasoning and five by means of how convincing the proof is. The participants were also given sets of arguments and asked to rank them from 1 (not a proof) to 4 (a proof). The teachers were able to identify all arguments that constituted proofs as proofs. However, a third of the ratings by the teachers deemed non-proof arguments as proofs. For example, Knuth (2002) found that 10 teachers decided that a proof of the converse of the statement:

"If $x > 0$, then $x + \frac{1}{x} \geq 2$."

was a proof of the statement. The researcher believed that the teachers focused on the algebraic manipulations rather than validity of the argument. He also found that four teachers found that an argument using a specific case to prove:

"If the sum of the digits of a whole number is divisible by 3, then the number itself is divisible by 3"

was a proof. However, he also found that many teachers found the specific case argument to be convincing enough to believe the statement to be true while understanding that it was not a proof. Therefore, he claimed that having empirical evidence provides more conviction in the truth of a statement. Knuth (2002) asserted that the teachers used mathematical criteria, such as proof method and mathematical reasoning, to validate arguments as proofs while using qualitative criteria, such as sufficient details and personal knowledge, to determine which proofs received higher ratings. He also found that teachers relied on specific examples, diagrams, familiarity with the argument, showing why it was true, and generality for conviction. With this, Knuth (2002) claimed that students' convictions do not rely on mathematical criteria, but rather personal criteria relating to the argument.

Stylianides and Stylianides (2009) conducted a study with 39 prospective elementary school teachers (K-6 grades). These students were mostly juniors with different undergraduate majors that had weak mathematical backgrounds. The researchers studied the process of "construction-evaluation" and how it relates to the understanding and evaluation of arguments. The students were asked to determine whether two conjectures were true or false with proof, and then were asked to explain whether they believed a mathematical proof was produced. The first conjecture was:

"The sum of any two consecutive odd numbers is a multiple of 4"

and the second conjecture was:

"If you multiply any odd number by 3 and then add 3, you get a multiple of 6."

Stylianides et al. (2009) found that all participants identified the two conjectures as true. For the first conjecture, 17 students either produced valid arguments that were not proofs or produced a proof. Of the remaining 22 students, 11 students were aware that their arguments were not proofs. Nine students produced empirical arguments with four believing that their argument was a mathematical proof. From the first conjecture to the second, the number of empirical arguments dropped from nine to three. Stylianides et al (2009) stated that students are often asked to present their ideas or solutions but are not asked whether they are satisfied with them or even believe them. The authors suggested that encouraging students to comment on their own work can help researchers and educators by showing students' understanding through explanations of their ideas rather than solely relying on their answers.

According to the results from Stylianides et al (2009), for the first conjecture, 22 students believed they produced a mathematical proof, but only nine actually did. Twelve students who produced arguments that were not proofs were aware that their work did not produce a proof. For the second conjecture, of the 27 students who believed they produced a mathematical proof, only 11 did. Eleven students who produced arguments that were not proofs were aware that their work did not produce a proof. Stylianides et al (2009) did not comment on these results, but these results warrant more research.

Weber (2010) conducted a study with 28 undergraduate mathematics majors in their sophomore or junior year who had already competed a proof course. The purpose of the study was to see which types of arguments undergraduate mathematics majors find convincing, which they consider to be proofs, and whether they have doubts about the claim. The students were given 10 arguments, one at a time, and were asked three questions relating to whether they understood and were convinced by the argument. The third question asked whether the student believed the argument was a rigorous proof. The students were asked to think aloud when answering each question. Two types of arguments were used in Weber's (2010) study, empirical and deductive arguments. There were seven deductive arguments (four being invalid) and one empirical argument. Weber (2010) found that only one student was completely convinced by the empirical argument, and only two students said it was a proof. He also found that the students judged logically flawed deductive arguments to be proofs 60% of the time. The author claimed that the students did this because they did not check the assumptions of the proof, did not check the mathematical principles used to deduce new assertions, lacked prior mathematical knowledge to catch errors, or simply overlooked the error. There was a total of 39 instances where participants said they were not convinced by an argument deemed to be a proof. Three quarters of the students were represented in these instances. Reasons for these unconvincing arguments included uncertainty about a step in the proof, inability to follow the flow of the proof, lack of personal conviction in the argument, and believing the proof should have been done a different way. Weber (2010) claimed that students did not take enough time to resolve these uncertainties which led them to disbelieve the proof. He also claimed that students use different criteria when determining conviction and validity in an argument.

Zaslavsky (2005) performed an experiment with preservice and in-service mathematics teachers who were also undergraduate students. He wished to improve learning of mathematics through tasks that bring about uncertainty. The author defines the term uncertainty as "a unifying term to encompass…conflict, doubt, and perplexity" (Zaslavsky, 2005, p. 300). There were three different kinds of uncertainty that a student may encounter. These were competing claims, unknown path or questionable conclusion, and non-readily verifiable outcomes. Competing claims were created when expectations create different viewpoints of the same thing. The uncertainty may have been caused by contradicting statements, misconceptions, expectations, or lack of conviction. An example of this phenomenon was the task asking students to determine the value of $(-8)^{1/3}$ and $(-8)^{2/6}$ where students say the former is -2 and the latter is 2. Unknown path or questionable conclusions were defined as the searching of a relationship that is unknown to the student who did not have an intuition about what is to be expected. Conjectures and questions about existences were common to this type of uncertainty as students may refute it or be required to explain and prove it. An example of this type of uncertainty was the task asking

> "For a given triangle ΔABC, is there a point D in the triangle such that the areas of triangles ΔABD, ΔACD, and ΔBCD are equal?"

Non-readily verifiable outcomes were generated by lack of confidence regarding the outcome or means of verification. The uncertainty may have been caused by the nature of the domain and difficulty in verifying the solutions. Zaslavsky (2005) claimed that the generation of uncertainty regarding mathematical tasks causes students to have a genuine need for proof and multiple sources of explanations.

Fischbein (1980) conducted a study regarding students' intuitions and convictions towards mathematics. He explained that convictions may be derived from formal argumentations, empirically inductive arguments, or intuitive intrinsic situations. Convictions derived from intuitive intrinsic situations occur when the representation of an argument feels valid and reliable. In addition to this, he stated that a student may be convinced by a proof but surprised by the statement. An example of this is the equivalence of the set of natural numbers and the set of positive even numbers. Statements may appear obvious to students but he or she may not have any capabilities to prove it – this occurrence generates conviction. Fischein (1980) claimed that once a student is satisfied by a statement, any additional proof does not strengthen conviction. He believed that students do not find formal, non-empirical, proofs convincing dueto their seemingly unproductive nature, instead, he believed that students relied on the productivity of calculations.

Alcock and Simpson (2004) conducted a study with 18 student volunteers from their first course involving formal proofs. Data were collected through semi-structured interviews. The researchers investigated the students' beliefs about their roles as learners of mathematics. They found that students able to visualize concepts were convinced by the correctness of their conclusions. Students that could not visualize concepts generally were not convinced because they did not fully understand the material. For students' beliefs about the learner's role, students were grouped by similarities. One group was created by students who had "an internal sense of authority," (Alcock and Simpson, 2004, p. 29) these students valued their own thinking and could decide whether they understand the concepts. This group of students could determine what was convincing using visual images and, when needed or requested, appropriate mathematical notation. Another group of students relied on personal intuition for conviction and found that detailed proofs were unrelated to establishing conviction. In general, these students separated their own conventions of providing convincing proof from formal mathematical proofs. The final group of students relied on external authority for evaluating work and progress, meaning that something is convincing if one told them it is true. Alcock and Simpson (2004) claimed that students who can visualize concepts are confident with their abilities to answer questions correctly and found mathematical proofs convincing. The less a student could visualize a concept, the worse he or she was at providing appropriate mathematical proof, and the source of conviction shifted from mathematics to non-mathematical sources.

A student's conviction in a proof and statement may help them appreciate math more. This may be done by allowing students to see gaps in knowledge, conviction, or notice redundancies in what is asked. Many of the studies mentioned previously did not analyze students' convictions regarding their proofs, only mathematical arguments presented to them. This study advances the knowledge on students' convictions with their proofs as well as their difficulties with specific topics, such as mathematical induction.

In the next chapter, the methods of data collection, research design, participants, timeline of this study, and methods of analysis are presented.

Chapter 3: Methods

In this chapter, the research design, participants, and instruments are described. The timeline for the study is discussed. The methods of analysis are presented for each research question.

Research Design and Participants

A study has been conducted with undergraduate students enrolled in a Discrete Mathematics course taught by the same instructor who agreed to participate in this study. The study was conducted at a four-year university in the United States. The undergraduate students' academic status ranged from freshmen to seniors as the course pre-requisite only required pre-calculus. Students who enrolled in a Discrete Mathematics course were STEM majors (e.g. mathematics, computer science, and mechanical engineering).

In the students' final exam for the course, there were three tasks relating to mathematical induction. These were created to probe students' difficulties with mathematical induction, their conviction in what they had proven, and what they found to be difficult. The tasks remained the same for the Spring 2019, Summer 2019, and Fall 2019 semesters; in the Fall 2019 semester, the third task was given more structure to provoke students to write about their difficulties for each previous task regarding mathematical induction. The instructor for the Discrete Mathematics courses used the textbook *Discrete Mathematics with Applications* (Epp, 2010) and followed Ernest's (1984) steps for mathematical induction laid out in Chapter 1.

Semi- structured interviews (Goldin, 2000) were conducted by the course instructor with all students from one section. During the interviews, students were asked to prove a theorem similar to proofs from class. Proofs by mathematical induction were randomly assigned to some of the interviewed students. Students were asked to think aloud to see how they attempt to prove statements using a method of proof. The interviews lasted approximately 15 minutes, were audio recorded and transcribed. The interview protocol is described in Appendix A and the induction tasks are presented in

Appendix ***C***.

All the undergraduate students enrolled in the Discrete Mathematics course taught the same instructor during the Spring 2019, Summer 2019, and Fall 2019 semesters, were asked to participate in the study. Seventy-eight undergraduate students 18 years old or older agreed to participate, out of a total of 102 students who took the course (76.5%).

Timeline

Table 1 describes the timeline of events for this study:

Table 1
Timeline for Actions

Time Period	Action
August 2018 – December 2019	Literature review of students' difficulties with proof by mathematical induction, students' convictions, and research methods
December 2018 – March 2019	Instrument development
April 2019	IRB Application approval to conduct research
May 2019	Recruitment of students in Spring 2019 semester
July 2019	Recruitment of students in Summer 2019 semester
December 2019	Proposal for thesis and recruitment of students in Fall 2019 semester
January 2020 – March 2020	Data analysis and thesis writing

From Table 1, it can be noted that the instruments were developed over the course of reviewing previous research and literature. IRB approval for the study (see

Appendix ***B***) was granted in April 2019 before obtaining students' consent for participation. Data collection began in May 2019, the spring semester of 2019, and continued for two additional semesters ending in December 2019, the fall semester of 2019. The data collected from the students' written answers to three tasks from the final exam and eight interviews were used to determine the students' difficulties and conviction with mathematical induction as a method of proof.

Instruments

The three tasks were created from several sources.

Task 1.

Task 1 is a common textbook problem and example (Rosen, 2012, p. 318; Epp, 2010, p. 257):

Prove by mathematical induction that

$1 + 2 + 4 + 8 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$, $n \geq 0$.

Task 2.

Task 2 is similar to a task in Epp (2010, p. 257). The domain was shifted to change the basis step from $n = 1$ to $n = 6$. The questions about the domain in the second task were changed from Stylianides et al. (2007), so students had to justify their answers in an open-ended format:

a) Prove the statement $P(n)$ by mathematical induction for $n \in \mathbb{N}$, $n \geq 6$,

$$P(n): 17 + 20 + 23 + \ldots + (3n - 1) = \frac{n(3n + 1)}{2} - 40.$$

b) Is the statement true for $n = 10$? Justify your answer.

c) Is the statement true for $n = 5$? Justify your answer.

Task 3.

Task 3 was altered from Stylianides et al. (2007) where students were asked to explain their thinking. Instead, students were asked to describe their difficulties:

> Refer to Tasks 1 and 2 only (proofs using mathematical induction). Comment on your difficulties, if any.

Methods of Analysis

The methods of analysis are presented for each research question.

Research Question 1.

Research Question 1 asked "What are the difficulties that undergraduate students have when using the method of mathematical induction, as described by Ernest's (1984) framework?"

To answer Research Question 1, Tasks 1 and 2a from the students' answers have been scored on a scale from 0 (lowest) to 10 (highest) using a rubric (see Table 2).

Table 2
Rubric for Task 1 and Task 2a

Steps with sub-steps of a proof by mathematical induction	Points
Basis step	1
Choosing the correct starting point, n_0	0.50
Correctly evaluating $P(n_0)$	0.50
Inductive step	7
Stating $P(k)$	1
Stating $P(k + 1)$	1
Substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$	1
Algebraic manipulations needed to show $P(k) \rightarrow P(k + 1)$	4
Writing the proof in the correct form	2
Invoking PMI	1

Stating the domain the introduced variable k	0.50
Explicitly writing that $P(k + 1)$ is to be shown assuming that $P(k)$ is true	0.50
Total	10

Table 2 shows each step of mathematical induction laid out by Ernest's (1984) framework: basis step, inductive step, and invocation of the Principle of Mathematical Induction.

The averages for each step of mathematical induction were calculated to see where students' actual difficulties may lie. The students' written responses to Task 1 and Task 2a were coded using the E-a framework described in Chapter 1. Table 3 shows how Ernest's (1984) framework was used to see the students' difficulties with Task 1 and Task 2a.

Table 3

Frequency of students' difficulties with Task 1 and Task 2a

Observed student difficulty by steps and sub-steps	Frequency (% out of 156)
Difficulties occuring in the basis step	
Omitting the basis step	-
Difficulty choosing the starting point, n_0	-
Difficulty evaluating $P(n_0)$	-
Difficulties occuring in the inductive step	
Difficulty stating $P(k)$	-
Difficulty stating $P(k + 1)$	-
Difficulty substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$	-
Difficulties with algebraic manipulations to show $P(k) \rightarrow P(k + 1)$	-
Difficulties with writing the proof in the correct form	
Did not invoke PMI	-

Not stating the domain for the introduced variable k	-
Not explicitly writing that $P(k + 1)$ is to be shown assuming that $P(k)$ is true	-

Table 3 presents the categories of students' difficulties with their proofs by mathematical induction and closely follows the rubric used to score Task 1 and Task 2a. The table provides the frequency of difficulties the students had for each step of a proof of mathematical induction.

Research Question 2.

Research Question 2 asked "How do students' self-described difficulties relate to how researchers describe students' difficulties with the method of mathematical induction?"

Students' written responses were coded using the E-a framework described in Chapter 1. Table 4 shows how the E-a framework was used to code students' difficulties using their responses to Task 3 and record the frequencies of occurrence of each difficulty.

Table 4

Frequency of students' self-described difficulties with Task 1 and Task 2a

Students' self-described difficulties by steps and sub-steps	Frequency (% out of 78)
Self-described difficulties in the basis step	
Omitting the basis step	-
Difficulty choosing the starting point, n_0	-
Difficulty evaluating $P(n_0)$	-
Self-desribed difficulties in the inductive step	
Difficulty stating $P(k)$	-
Difficulty stating $P(k + 1)$	-
Difficulty substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$	-
Difficulties with algebraic manipulations to show $P(k) \rightarrow P(k + 1)$	-

Self-described difficulties with writing the proof in the correct form	
Did not invoke PMI	-
Not stating the domain for the introduced variable k	-
Not explicitly writing that $P(k + 1)$ is to be shown assuming that $P(k)$ is true	-

Table 3 and Table 4 were compared to determine whether the frequencies of student difficulties as described by the researcher in this study matches the students' descriptions on their difficulties.

Research Question 3.

Research Question 3 asked "What are students' convictions in mathematical induction as a method of proof?"

Written Task 2b and Task 2c were administered as extensions of Task 2a to determine the students' conviction in their proofs. Students' written answers to Task 2b and 2c were analyzed using the WMRe framework described in Chapter 1. Table 5 shows the categories of the WMRe framework.

Table 5
Students' convictions and disbeliefs in Task 2a

	Conviction		Disbelief		
	Proof	Absolute	Unconvinced	Empirical	Total
Task 2b	-	N/A	-	-	-
Task 2c	-	-	N/A	-	-

Table 5 shows that students' answers were placed in one of the categories of the WMRe framework described in Chapter 1. Absolute Conviction was omitted from Task 2b because the task asks about a value within the domain of the proven statement. Unconvinced Disbelief was omitted from Task 2c because the task asks about a value outside the domain of the proven statement.

Table 6 shows the interaction between the responses of Task 2b and Task 2c using the WMRe categories. The interaction between the responses of Task 2b and Task 2c is necessary because it shows how students change the way they establish conviction depending on the kind of question asked (e.g. whether $P(n)$ is true for n within or not within the proven domain of $P(n)$).

Table 6
Students' interactions of responses between Task 2b and Task 2c

	Task 2b result			
Task 2c result	Proof conviction	Unconvinced disbelief	Empirical disbelief	Total
Proof conviction	-	-	-	-
Absolute conviction	-	-	-	-
Empirical disbelief	-	-	-	-
Total	-	-	-	-

In Table 6, each entry in the table, apart from the totals, shows the frequencies of responses for both Task 2b and Task 2c. For example, the first entry in the table was used to show the number of students' answers that were categorized as proof conviction for Task 2b while also categorized as proof conviction for Task 2c.

Summary

In this chapter, the research design was discussed along with the background for the participants. The timeline for the study was presented. The instruments, comprised of three tasks, were presented and the methods of analysis for each research question were discussed. In the next chapter, the data analyses are presented.

Chapter 4: Data Analyses

Introduction

In this chapter, data analyses are presented, by research question. Data collected have been analyzed using the E-a framework for Research Questions 1 and 2, and the WMRe framework for Research Question 3. These frameworks have been described in Chapter 1.

Research Question 1

Research Question 1 asked "What are the difficulties that undergraduate students have when using the method of mathematical induction, as described by Ernest's (1984) framework?"

Using the E-a framework described in Chapter 1 to analyze students' difficulties with mathematical induction, the overall students' performances on Task 1 and Task 2a have been analyzed by calculating the mean score for each using the steps of mathematical induction (see Table 2 in Chapter 3).

Inter-rater agreement.

The inter-rater agreement analyses (Cohen, 1960) were calculated using the Kappa statistic to determine the reliability between the two raters. The Kappa statistic, k, is defined by:

$$k = \frac{p_0 - p_e}{1 - p_e}$$

where p_0 is the proportion of units which the raters agreed on the rating, and p_e is the proportion of units which is expected that the raters agree by chance (Cohen, 1960). Twenty students' responses to Task 1 and Task 2a were scored by the researcher and a second rater. For Task 1, the agreement between the two raters was 0.77 ($p < 0.01$) with CI (0.57, 0.97). After discussing the differences and disagreements, the rating between the raters came to be 1.00 ($p < 0.01$). The reason for differences was due to sufficient work regarding manipulations of terms with exponents in showing $P(k) \rightarrow P(k + 1)$. For Task 2a, the agreement between the two raters was 0.94 ($p < 0.01$) with CI (0.82, 1.06).

Student performance on Task 1 and Task 2a.

The students' performance was determined by scoring each student's response to Task 1 and Task 2a (see Table 7).

Table 7
Mean scores for Task 1 and Task 2a

Steps and sub-steps of a proof by mathematical induction	Task 1	Task 2a
Basis step	0.88 (88%)	0.84 (84%)
Choosing the correct starting point, n_0	0.44 (88%)	0.45 (90%)
Correctly evaluating $P(n_0)$	0.44 (88%)	0.39 (78%)
Inductive step	6.06 (86%)	5.79 (83%)
Stating $P(k)$	0.82 (82%)	0.69 (69%)
Stating $P(k + 1)$	0.94 (94%)	0.74 (74%)
Substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$	0.87 (87%)	0.82 (82%)
Algebraic manipulations needed to show $P(k) \rightarrow P(k + 1)$	3.43 (86%)	3.54 (89%)
Writing the proof in the correct form	1.09 (55%)	0.92 (46%)
Invoking PMI	0.48 (48%)	0.44 (44%)
Stating the domain for the introduced variable k	0.19 (37%)	0.14 (28%)
Explicitly writing that $P(k + 1)$ is to be shown assuming that $P(k)$ is true	0.42 (84%)	0.34 (68%)
Total	7.01 (78%)	6.69 (74%)

Table 7 shows that the students' performance was below the maximum number of points for each step and sub-step of mathematical induction. This means that students experienced difficulties at each step of mathematical induction.

Table 8 follows the E-a framework described in Chapter 1. It is populated by the number of occurrences in each category of difficulty.

Table 8
Frequencies of students' difficulties with Task 1 and Task 2a

Observed student difficulty by steps and sub-steps	Frequency (% out of 156)
Difficulties occuring in the basis step	
Omitting the basis step	5 (3%)
Difficulty choosing the correct initial value, n_0	3 (2%)
Difficulty evaluating $P(n_0)$	4 (3%)
Difficulties occuring in the inductive step	
Difficulty stating $P(k)$	31 (20%)
Difficulty stating $P(k + 1)$	55 (35%)
Difficulty substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$	19 (12%)
Difficulties with algebraic manipulations to show $P(k) \rightarrow P(k + 1)$	32 (21%)
Difficulties with writing the proof in the correct form	
Did not invoke PMI	29 (19%)
Not stating the domain of the introduced variable k	60 (38%)
Not explicitly writing that $P(k + 1)$ was to be shown assuming $P(k)$ was true	90 (58%)

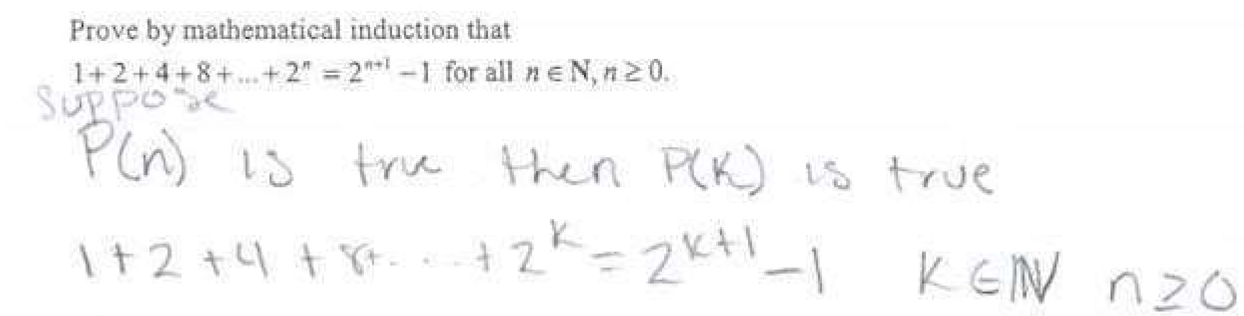
Table 8 shows the frequency of instances that a difficulty occurred. There was a total of 156 instances, 78 student responses for two tasks. The number in parentheses is the percent out of 156 total possible instances. The table shows that the researcher observed difficulties that occurred at each step of mathematical induction.

Difficulties with the basis step for Task 1 and Task 2a.

The students experienced difficulties with the basis step as seen from Table 7 with a mean score of 0.86 (86%) between Task 1 and Task 2a.

Students were able to complete the basis step frequently resulting in a high mean score of 0.86 out of 1. Nearly every student was able to complete the basis step regardless of producing a proof.

From Table 8, there were five instances, 3% of the responses, where students omitted the basis step. Figure 3 is an example of a student response that omitted the basis step from the proof by mathematical induction.


Prove by mathematical induction that
$1+2+4+8+...+2^n = 2^{n+1}-1$ for all $n \in N, n \geq 0$.

Suppose
$P(n)$ is true then $P(K)$ is true
$1+2+4+8...+2^K = 2^{K+1}-1 \quad K \in \mathbb{N} \quad n \geq 0$

Figure 3. Example of a student's answer omitting the basis step from Task 1.

Figure 3 shows that the proof by mathematical induction began with the inductive step by stating $P(k)$ and stating the domain for the introduced variable k, meaning that the basis step was omitted from the proof by mathematical induction.

As seen from Table 8, there were three instances, 2% of the responses, where students incorrectly chose the starting point, n_0, for their proofs by mathematical induction. Figure 4 is an example of a student response that had an incorrect starting point, n_0.

Prove by mathematical induction that $1+2+4+8+...+2^n = 2^{n+1}-1$ for all $n \in \mathbf{N}, n \geq 0$.

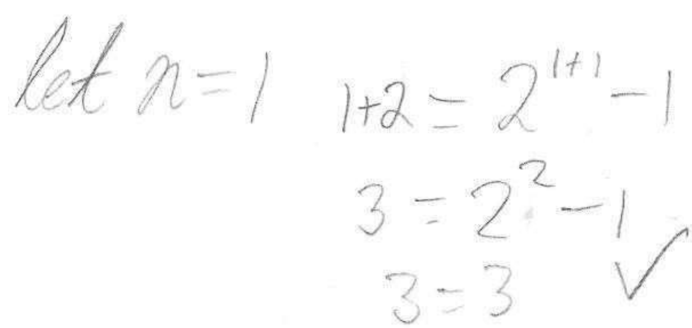


Figure 4: Example of difficulty choosing the starting point, n_0, for Task 1.

Figure 4 shows that the student began the proof by mathematical induction of the statement, $P(n)$, in Task 1 with $P(1)$ instead of $P(0)$.

As seen from Table 8, there were four instances, 3% of the responses, where students had difficulty calculating $P(n_0)$. This explains, given this category of students' answers, the reason why the inductive step and invocation of PMI mean scores are much lower than the basis step mean score. An example of a student response having difficulty with evaluating $P(n_0)$ can be seen in Figure 5.

a) Prove the statement P(n) by mathematical induction for $n \in \mathrm{N}$, $n \geq 6$,
$P(n): 17+20+23+...+(3n-1) = \frac{n(3n+1)}{2} - 40$.

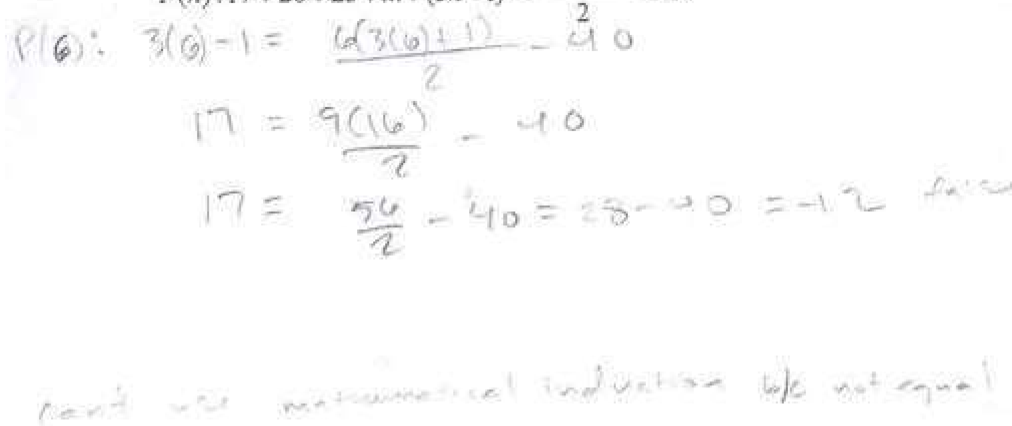


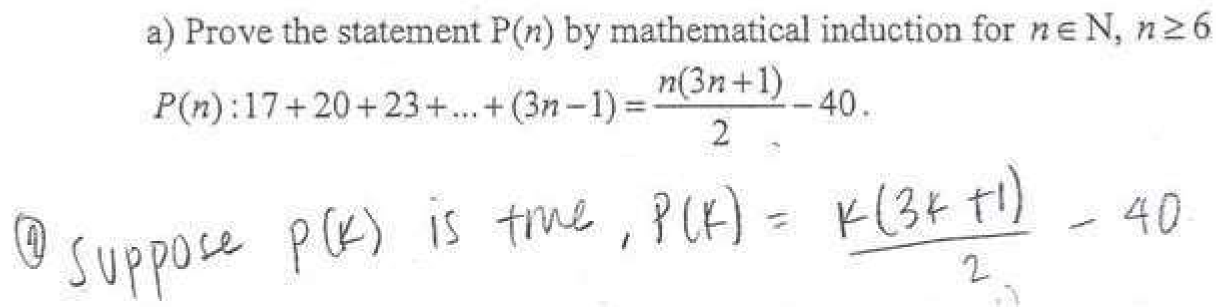
Figure 5. Example of difficulty with evaluating $P(n_0)$ for Task 2a.

Figure 5 shows that the beginning value was correctly chosen for the basis step, $n = 6$ in this case, for the proof by mathematical induction. However, $P(6)$ was miscalculated to get 17 = -12, which is not true. The student then claims that mathematical induction cannot be used because $P(6)$ turned out to be not true, according to the calculation.

Difficulties with the inductive step for Task 1 and Task 2a.

The student experienced difficulties with the inductive step of their proofs by mathematical induction. This can be seen from Table 7 where the students had a mean score of 5.93 (85%) out of 7 for the entire inductive step between Task 1 and Task 2a.

From Table 8, there were 31 instances where students had difficulties stating $P(k)$. Figure 6 is an example of a student response that showed difficulties with stating $P(k)$.


a) Prove the statement P(n) by mathematical induction for $n \in \mathrm{N}$, $n \geq 6$

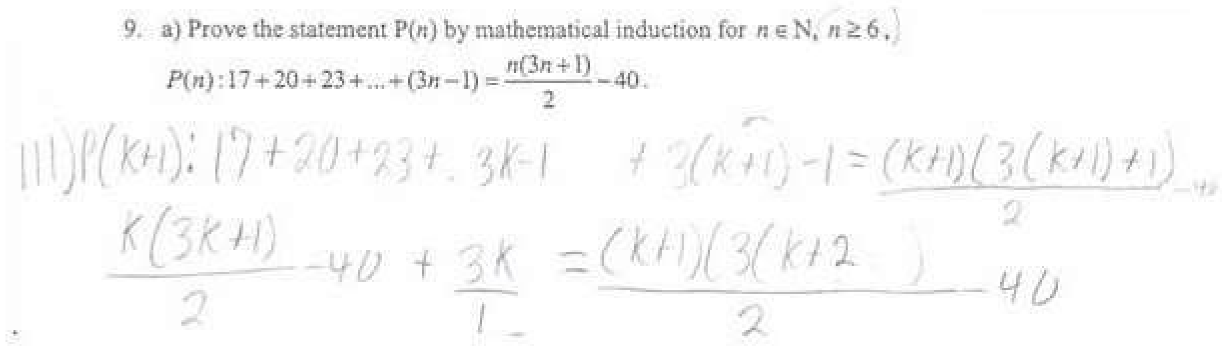
$P(n): 17+20+23+\ldots+(3n-1)=\frac{n(3n+1)}{2}-40$.

① Suppose $P(k)$ is true, $P(k) = \frac{k(3k+1)}{2} - 40$

Figure 6. Example of difficulty writing $P(k)$ for Task 2a.

Figure 6 shows that $P(k)$ was not stated correctly. In this case, only the right side of the identity of the statement was used to define $P(k)$.

As seen from Table 8, there were 55 instances where students did not state $P(k + 1)$ correctly. Figure 7 is an example of a student response that did not state $P(k + 1)$ correctly.


9. a) Prove the statement P(n) by mathematical induction for $n \in \mathrm{N}$, $n \geq 6$.

$P(n): 17 + 20 + 23 + \ldots + (3n-1) = \frac{n(3n+1)}{2} - 40$.

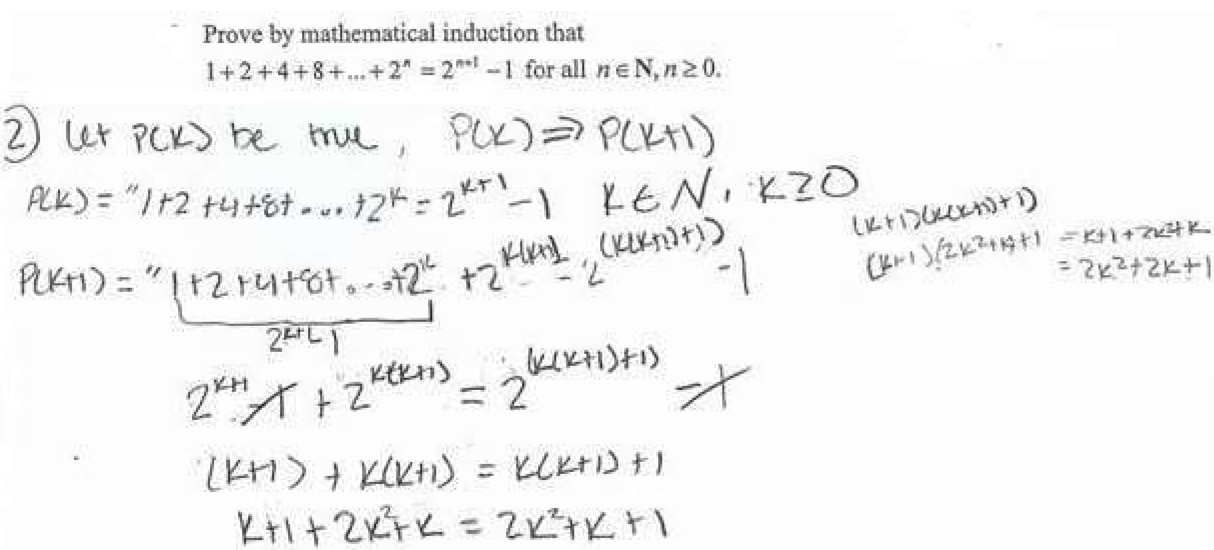
(ii) $P(k+1): 17+20+23+ . \; 3k-1 \quad + 3(k+1)-1 = \frac{(k+1)(3(k+1)+1)}{2} - 40$

$\frac{k(3k+1)}{2} - 40 + \frac{3k}{1} = \frac{(k+1)(3(k+2\;\;))}{2} - 40$

Figure 7. Example of difficulty stating $P(k + 1)$ in Task 2a.

Figure 7 shows that state $P(k + 1)$ was not stated correctly. The response shows that 1 was added to the terms with k on both sides of the statement.

As seen from Table 8, there were 32 instances where students had difficulties with the algebraic manipulations needed to prove $P(k) \rightarrow P(k + 1)$. Figure 8 is an example of a student response that showed difficulties with the algebraic manipulations needed to prove that $P(k)$ implies $P(k + 1)$.


Prove by mathematical induction that

$1 + 2 + 4 + 8 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \in \mathrm{N}, n \geq 0$.

2) Let P(K) be true, P(K) ⇒ P(K+1)

$P(K) = "1+2+4+8+\ldots+2^k = 2^{k+1}-1 \quad K \in N, K \geq 0$

$P(K+1) = "\underbrace{1+2+4+8+\ldots+2^k}_{2^{k+1}-1} + 2^{k(k+1)} = 2^{(k(k+1)+1)} - 1$

$2^{k+1} \cancel{-1} + 2^{k(k+1)} = 2^{(k(k+1)+1)} \cancel{-1}$

$(k+1) + k(k+1) = k(k+1)+1$

$k+1+2k^2+k = 2k^2+k+1$

$(k+1)(k(k+1)+1)$

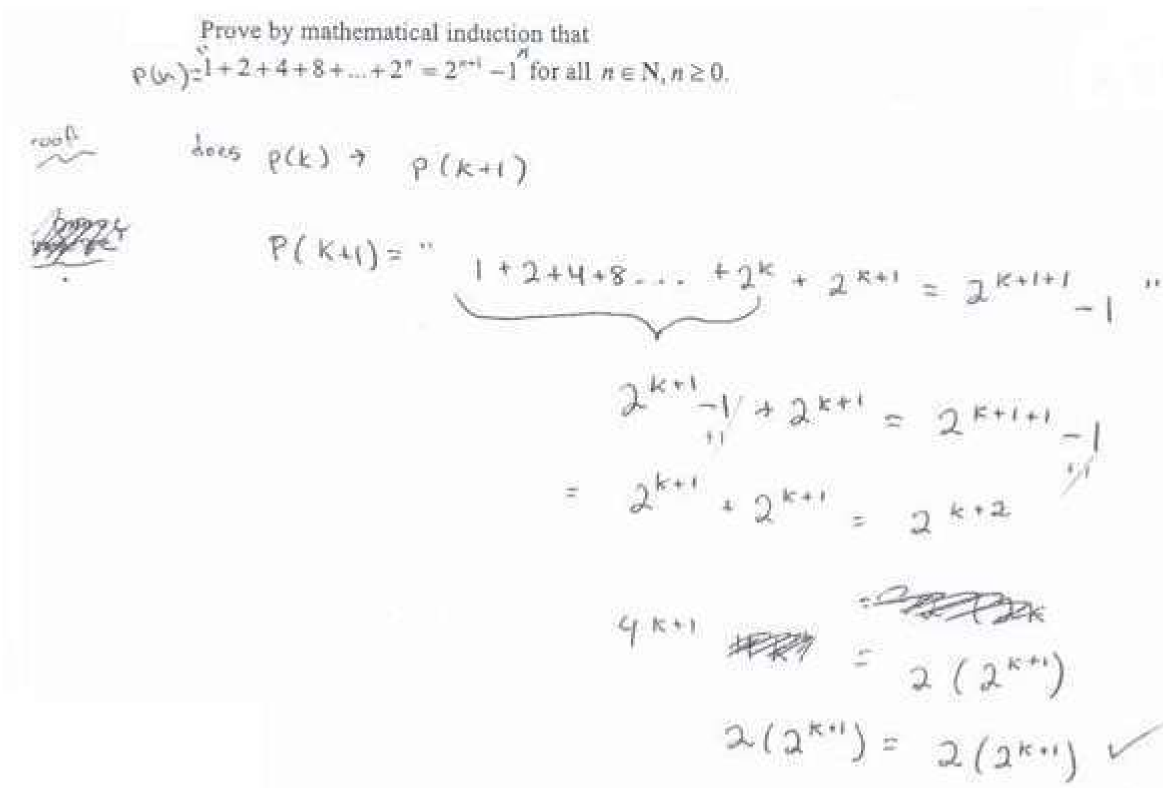
$(k+1)(2k^2+k)+1 = k+1+2k^2+k$

$= 2k^2+2k+1$

Figure 8. Example of difficulties with showing $P(k) \rightarrow P(k + 1)$ in Task 1.

Figure 8 shows that difficulties were experienced after the substitution of $P(k)$ into $P(k + 1)$. After this substitution, only the exponents were used and they were simply added to check if they were equal – the exponents were then treated as polynomials, no longer as exponents.

Students sometimes experienced difficulties with manipulating the terms to prove the implication statement, $P(k) \rightarrow P(k + 1)$. Figure 9 is an example of a student response that showed difficulty in proving $P(k) \rightarrow P(k + 1)$.


Prove by mathematical induction that
$P(n)$: $1+2+4+8+\dots+2^n = 2^{n+1}-1$ for all $n \in \mathbb{N}, n \geq 0$.

proof

does $P(k) \rightarrow P(k+1)$

$P(k+1) =$ " $\underbrace{1+2+4+8\dots+2^k} + 2^{k+1} = 2^{k+1+1} - 1$ "

$2^{k+1} - 1 + 2^{k+1} = 2^{k+1+1} - 1$

$= 2^{k+1} + 2^{k+1} = 2^{k+2}$

$4^{k+1} = 2(2^{k+1})$

$2(2^{k+1}) = 2(2^{k+1})$ ✓

Figure 9: Example of difficulty manipulating terms to show $P(k) \rightarrow P(k + 1)$ for Task 2a.

Figure 9 shows that $2^{k+1} + 2^{k+1} = 4^{k+1}$ was written after $P(k)$ was substituted into $P(k + 1)$, however, this equality does not hold. The terms were manipulated an equality was represented that held true.

Difficulties with writing the proof in the correct form for Task 1 and Task 2a.

Students performed poorly on the conclusions of their proofs where they needed to invoke PMI. This can be seen from Table 7 as the mean score was 1 out of 2 (50%) between the tasks.

As seen from Table 8, there were 60 instances where students did not state the domain for the introduced variable *k*. Figure 10 is an example of a student response that did not state the domain for an introduced variable.

a) Prove the statement P(n) by mathematical induction for $n \in \mathrm{N}$, $n \geq 6$,

$$P(n): 17+20+23+\ldots+(3n-1)=\frac{n(3n+1)}{2}-40.$$

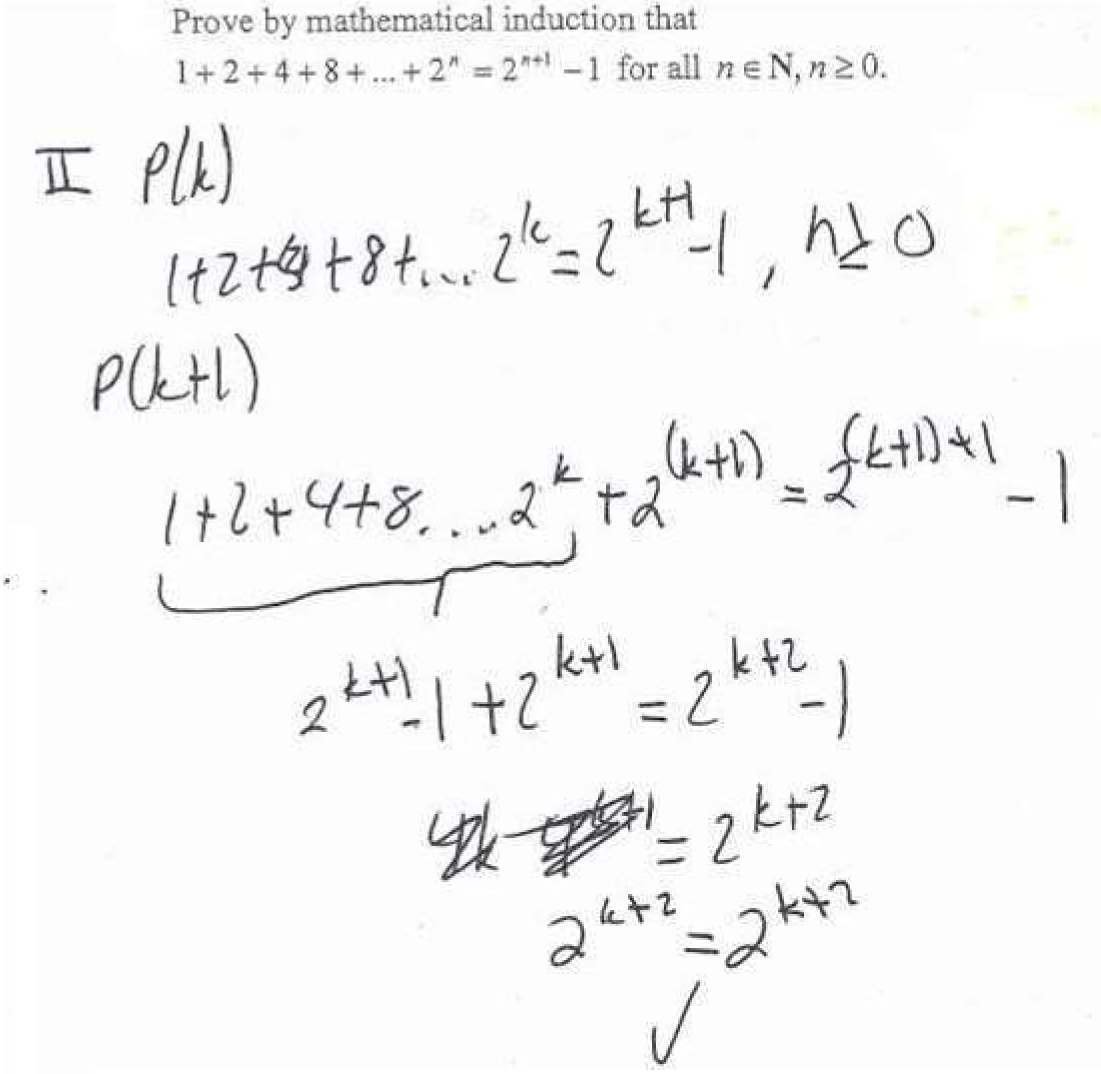
$$P(k) = 17+20+23+\ldots+(3k-1) = \frac{k(3k+1)}{2} - 40$$

$$P(k+1) = \underbrace{17+20+23+\ldots+(3k-1)}_{\frac{k(3k+1)}{2}-40} + (3(k+1)-1) = \frac{(k+1)(3(k+1)+1)}{2} - 40$$

Figure 10. Example of not stating the domain for the introduced variable *k* for Task 2a.

Figure 10 shows that the variable *k* was introduced when performing the inductive step but the domain for it was not stated.

As seen from Table 8, there were 45 instances where students did not write the implication statement, $P(k) \rightarrow P(k+1)$. Figure 11 is an example of a student response that did not have the implication $P(k) \rightarrow P(k+1)$ and did not assume that $P(k)$ was true.



Prove by mathematical induction that
$1 + 2 + 4 + 8 + ... + 2^n = 2^{n+1} - 1$ for all $n \in \mathrm{N}, n \geq 0$.

II $P(k)$

$1+2+4+8+\ldots 2^k = 2^{k+1}-1, \; n \geq 0$

$P(k+1)$

$\underbrace{1+2+4+8\ldots 2^k} + 2^{(k+1)} = 2^{(k+1)+1} - 1$

$2^{k+1}-1+2^{k+1} = 2^{k+2}-1$

$= 2^{k+2}$

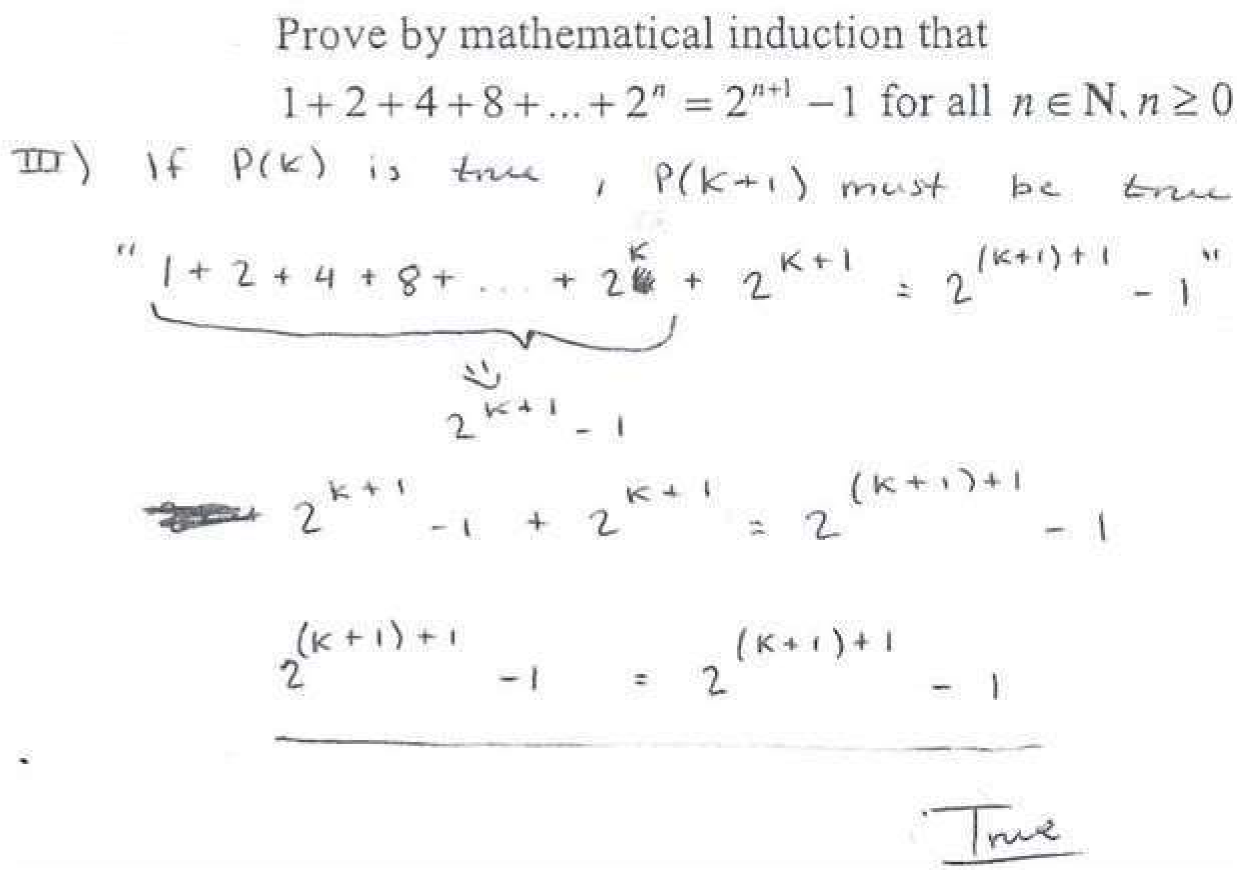
$2^{k+2} = 2^{k+2}$

✓

Figure 11. Example of neither writing the implication statement, $P(k) \rightarrow P(k + 1)$ nor assuming $P(k)$ to be true for Task 1.

Figure 11 shows that the implication, $P(k) \rightarrow P(k + 1)$, was not explicitly written because it did not appear in the proof. Also, the assumption that $P(k)$ was true did not appear in the proof.

As seen from Table 8, there were 29 instances where students did not invoke PMI. Instead, a check mark or "true" was used to act as the end of the proofs. Figure 12 is an example of a student response that had "true" to conclude the proof by mathematical induction. Figure 13 is an example of a student response that had a check mark to conclude the proof by mathematical induction. In both of these figures, the invocation of PMI is omitted.



Prove by mathematical induction that

$1+2+4+8+...+2^n = 2^{n+1}-1$ for all $n \in \mathrm{N}, n \geq 0$.

III) If P(k) is true, P(k+1) must be true

"$\underbrace{1+2+4+8+\dots+2^{k}}_{2^{k+1}-1} + 2^{k+1} = 2^{(k+1)+1} - 1$"

$2^{k+1} - 1 + 2^{k+1} = 2^{(k+1)+1} - 1$

$2^{(k+1)+1} - 1 = 2^{(k+1)+1} - 1$

True

Figure 12: Example 1 of answer missing the invocation of PMI for Task 1.

Figure 12 shows that the invocation of PMI was omitted. Instead, “true” was written at the end of the inductive step acknowledging the equality that $P(k)$ implies $P(k + 1)$ thus ending the proof.

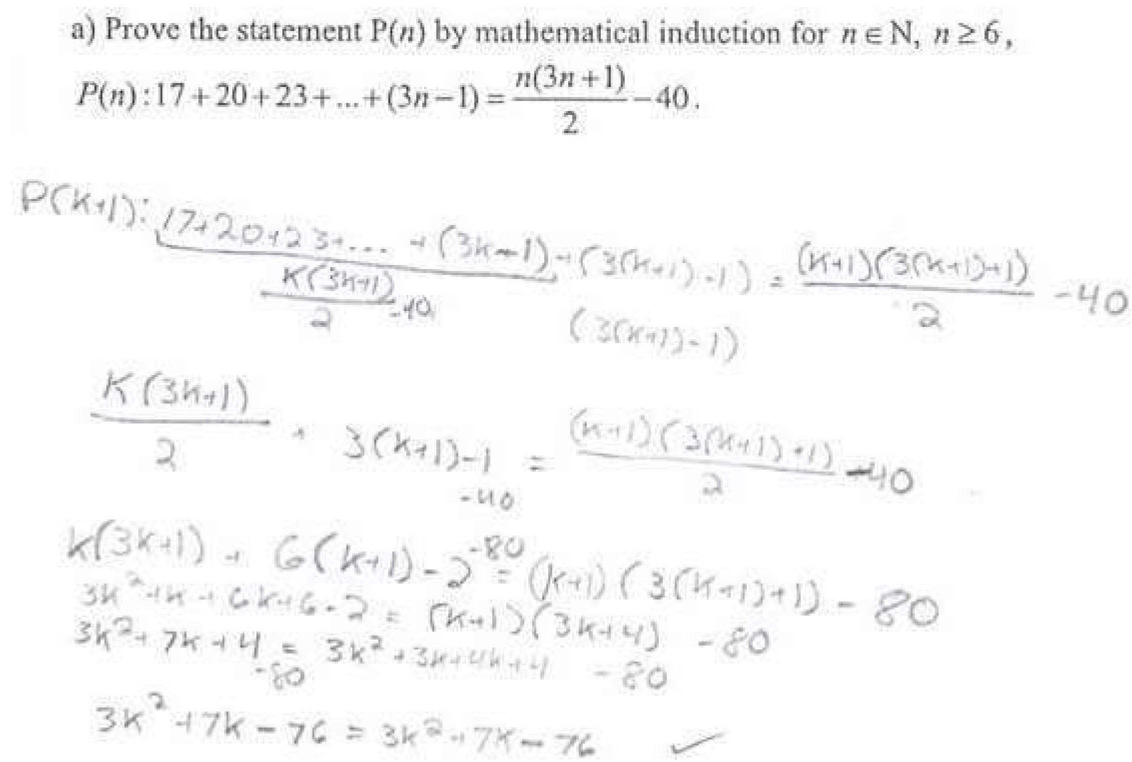


Figure 13. Example 2 of answer missing the invocation of PMI for Task 2a.

Figure 13 shows that the invocation of PMI was omitted. Instead, a check mark acknowledging that $P(k)$ implies $P(k + 1)$ was used to conclude the proof.

In some cases, students stated that something other than $P(n)$ was proven in Task 2a. Figure 14 is an example of a student response that stated something besides $P(n)$ was proven by PMI.

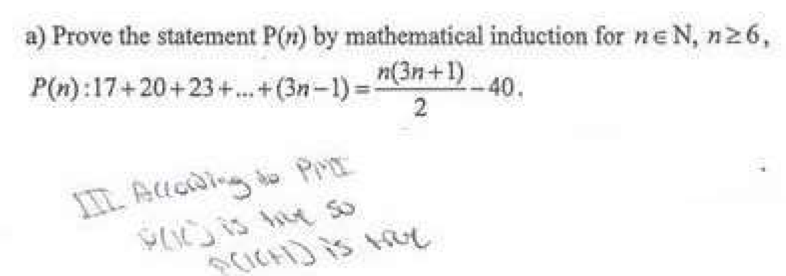


Figure 14. Example of difficulty with PMI for Task 2a.

Figure 14 shows that the statement, $P(k + 1)$ in Task 2a, was proven by PMI and is therefore true.

It can be seen from Table 7 that students encountered difficulties at each step of mathematical induction. The accompanying figures for the difficulties support this claim. The most numerous instances where difficulties were encountered writing the proof in the correct form of their proofs by mathematical induction.

Summary of analysis of students' written responses to Task 1 and Task 2a.

Regarding students' written responses, students experienced difficulty at each step of a proof by mathematical induction (see Table 7 and Table 8).

Task 2a was more difficult for students with a mean score of 6.69 out of 10 while Task 1 with a mean score of 7.01. There was a total of 156 student responses, 78 responses for each of the two tasks Task 1 and Task 2a. Students performed better in the basis step, with a mean score of 0.88 out of 1 for Task 1, and 0.84 out of 1 for Task 2a.

Many difficulties occurred in the inductive step, although their performance on this step was good (mean score of 6.06 out of 7 for Task 1 and a mean score of 5.79 out of 7 for Task 2a). The frequency for the difficulty for stating $P(k)$ was 31 out of 156 responses (20% of the responses). The frequency of difficulty for stating $P(k + 1)$ was 55 out of 156 responses (35% of the responses). The frequency for the difficulty with substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$ was found in 19 responses out of 156 (12% of the responses). Students experienced difficulties with the algebraic manipulation of the terms to prove $P(k) \rightarrow P(k + 1)$; this difficulty had a frequency of 32 out of 156 (21% of the responses).

The most difficult step for students was writing the proof in the correct form (mean scores of 1.09 out of 2, and 0.92 out of 2 for Task 1, and Task 2a, respectively). Out of 156 responses, in 90 of them (58%) students did not explicitly write the implication statement $P(k) \rightarrow P(k + 1)$ and assumed $P(k)$ to be true. Students did not state the domain for the introduced variable k in 60 of the 156 responses (38%). Omitting to invoke PMI had a frequency of 29 responses out of 156 (19%).

Next, the analysis of the difficulties with mathematical induction tasks from the interviews with students is presented.

Students' difficulties with mathematical induction from interviews.

The interviews conducted with eight students were used to further describe student difficulties with mathematical induction. Using the E-a framework for difficulties with mathematical induction (see Figure 1), the researcher and second rater coded the students' answers from interviews. It should be noted these interviews took place before the final exam.

Inter-rater reliability.

The inter-rater agreement analyses (Cohen, 1960) were calculated using the Kappa statistic to determine the reliability between the two raters. The first round of rating resulted in a Kappa statistic of 0.67 ($p < 0.01$) with CI (0.49, 0.85); after discussing the disagreements and differences, the second round of rating resulted in a Kappa statistic of 1.00 ($p < 0.01$). The reason for differences was that students often wrote their statements, $P(k)$ and $P(k + 1)$, without labelling them, and one rater considered it a difficulty while the other did not. This was resolved by not considering the non-labelling of statement as a difficulty because the students were able to complete their proofs by mathematical induction. After discussing the disagreements and differences, the second round of rating resulted in a Kappa statistic of 1.00 ($p < 0.01$).

Student difficulties with induction tasks during the interviews.

Table 9 shows the frequency for the students' difficulties with the tasks during the interviews.

Table 9
Frequencies of difficulties for tasks from interview

Observed difficulty at steps and sub-steps of mathematical induction	Frequency
Difficulties occuring in the basis step	
Omitting the basis step	2
Difficulty choosing the correct starting point, n_0	2
Difficulty evaluating $P(n_0)$	1
Difficulties occuring in the inductive step	
Difficulty stating $P(k)$	3
Difficulty stating $P(k + 1)$	3
Difficulty substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$	3
Difficulties with algebraic manipulations to show $P(k) \rightarrow P(k + 1)$	3
Difficulties with writing the proof in the correct form	
Did not invoke PMI	5
Not stating the domain for the introduced variable k	7
Not explicitly writing that $P(k) \rightarrow P(k + 1)$ was to be shown assuming that $P(k)$ is true	4

From the interviews, the students showed they had difficulties with each step of mathematical induction.

Student difficulty by omitting the basis step.

Figure 15 is an example of a student who omitted the basis step in his or her proof by mathematical induction during the interview.

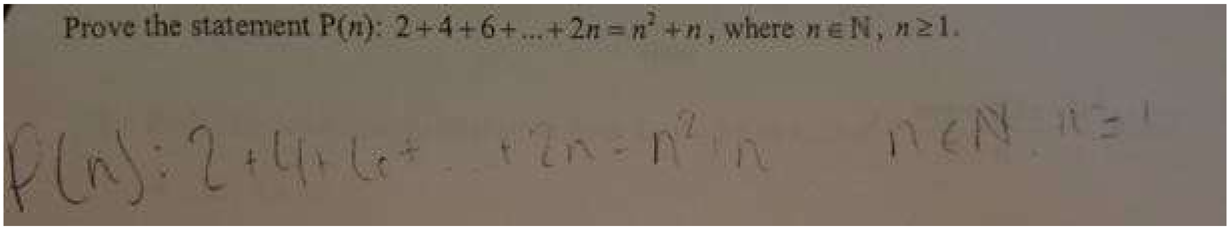


Figure 15. Example of answer without the basis step.

Figure 15 shows that the student began by stating $P(n)$ in the inductive step instead of verifying $P(1)$.

Student difficulty with choosing n_0 and evaluating $P(n_0)$.

Figure 16 is an example of a student who chose the wrong n_0 and evaluated $P(n_0)$ incorrectly for the basis step in his or her proof by mathematical induction during the interview.

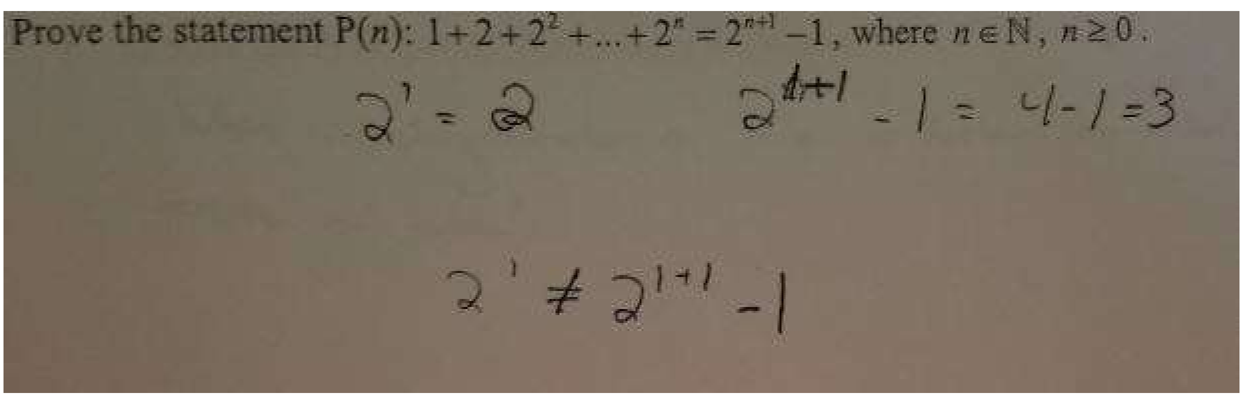


Figure 16. Example of difficulty with choosing n_0 and evaluating $P(n_0)$.

Figure 16 shows that the student chose $n_0 = 1$ instead of choosing $n_0 = 0$ for the basis step in his or her proof by mathematical induction during the interview. This caused the student to see that the statement was not true because $P(n_0)$ was incorrectly evaluated.

Student difficulty with stating P(k).

The task given to the student during his or her interview was:

Prove the statement $P(n)$: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$, where $n \in \mathbb{N}$, $n \geq 1$.

The following is an excerpt from the interview with one student who experienced difficulties with stating $P(k)$.

Student: For every question that pops up for mathematical induction what we do is basically we consider that for n equals k is true, we just assume that if n equals k, then it is true. So, we just write it down like that… (*writes* "$P(n) = \frac{k}{k+1}$").

From the excerpt, the student had difficulty with stating $P(k)$ because he or she only used the sum, one side of the equality. Figure 17 is the student's written work from the excerpt.

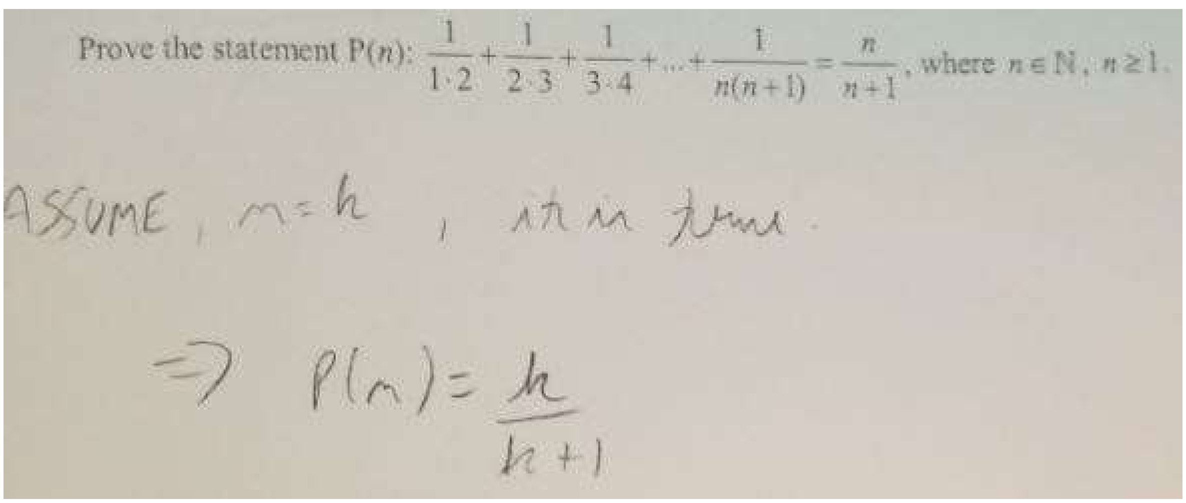


Figure 17. Example of difficulties with stating $P(k)$.

Figure 17 shows that the student stated $P(k)$ incorrectly by only using one side of the equality of the statement, in this case the student only used the sum.

Student difficulty with stating P(k + 1).

The task given to the student during his or her interview was:

Prove the statement $P(n)$: $2 + 4 + 6 + \ldots + 2n = n^2 + n$, where $n \in \mathbb{N}$, $n \geq 1$.

The student knew to assume $P(k)$ to be true and show that $P(k) \rightarrow P(k)$. The following is an excerpt from the interview:

> Student: I'm not completely sure on how to do $P(k)$, I can't remember, but I know $P(k + 1)$ is for sure … $n^2 + n + 1$. I think that would give you $P(k + 1)$ because we assumedthat $P(k)$ is true. I'm trying to remember….

From the excerpt, the student does not comment on the error in stating $P(k + 1)$. Figure 18 is the student's written work from the interview.

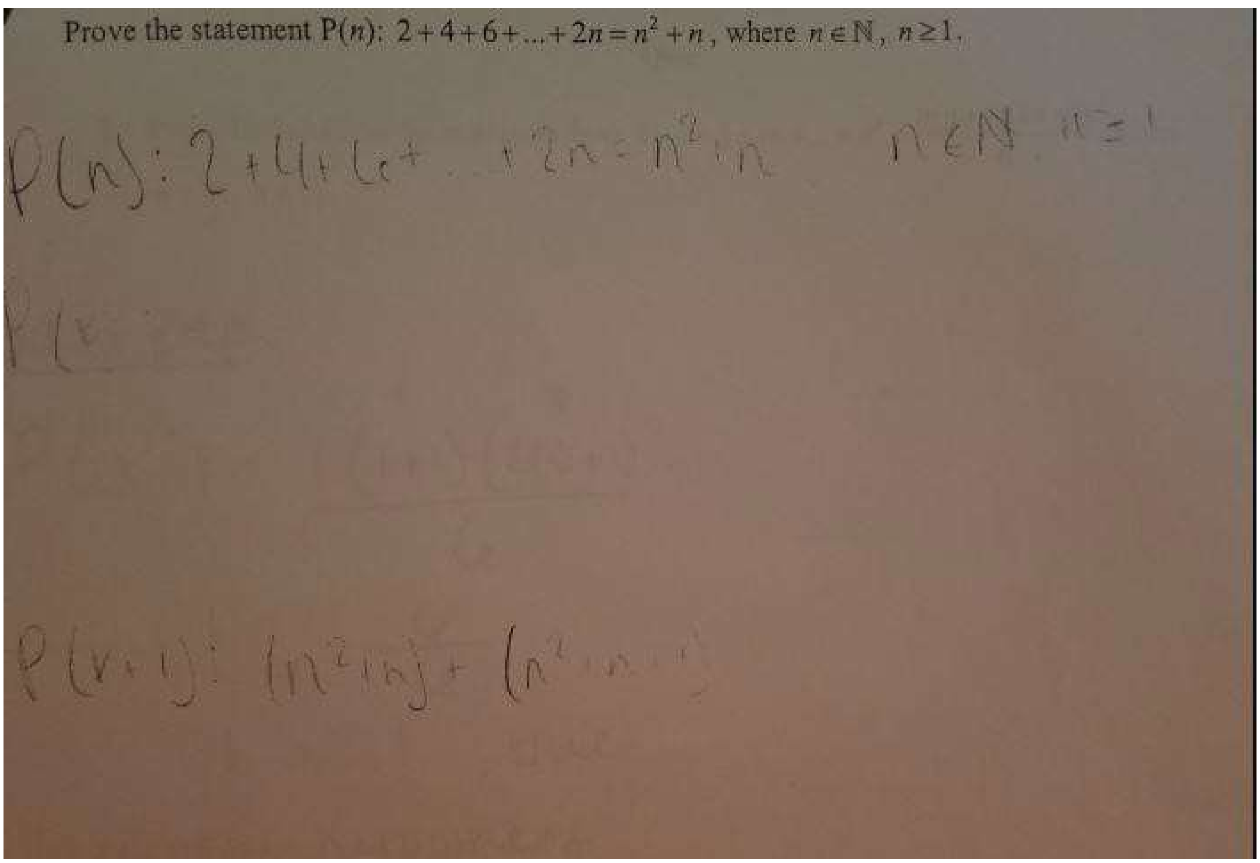


Figure 18. Example of difficulty with stating $P(k + 1)$.

Figure 18 shows that the student stated $P(k + 1)$ as $(n^2 + n) + (n^2 + n + 1)$ which is incorrect.

Student difficulty with substituting P(k) into P(k + 1) and showing P(k) → P(k + 1).

There were three instances (38%) where students had difficulties with substituting $P(k)$ into $P(k + 1)$. There were also 3 instances where students had difficulties with the algebraic manipulations $P(k) \rightarrow P(k + 1)$. Figure 19 is an example of a student who had difficulty with both of these steps during his or her interview. The task was:

Prove the statement $P(n)$: $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$, where $n \in \mathbb{N}$, $n \geq 1$.

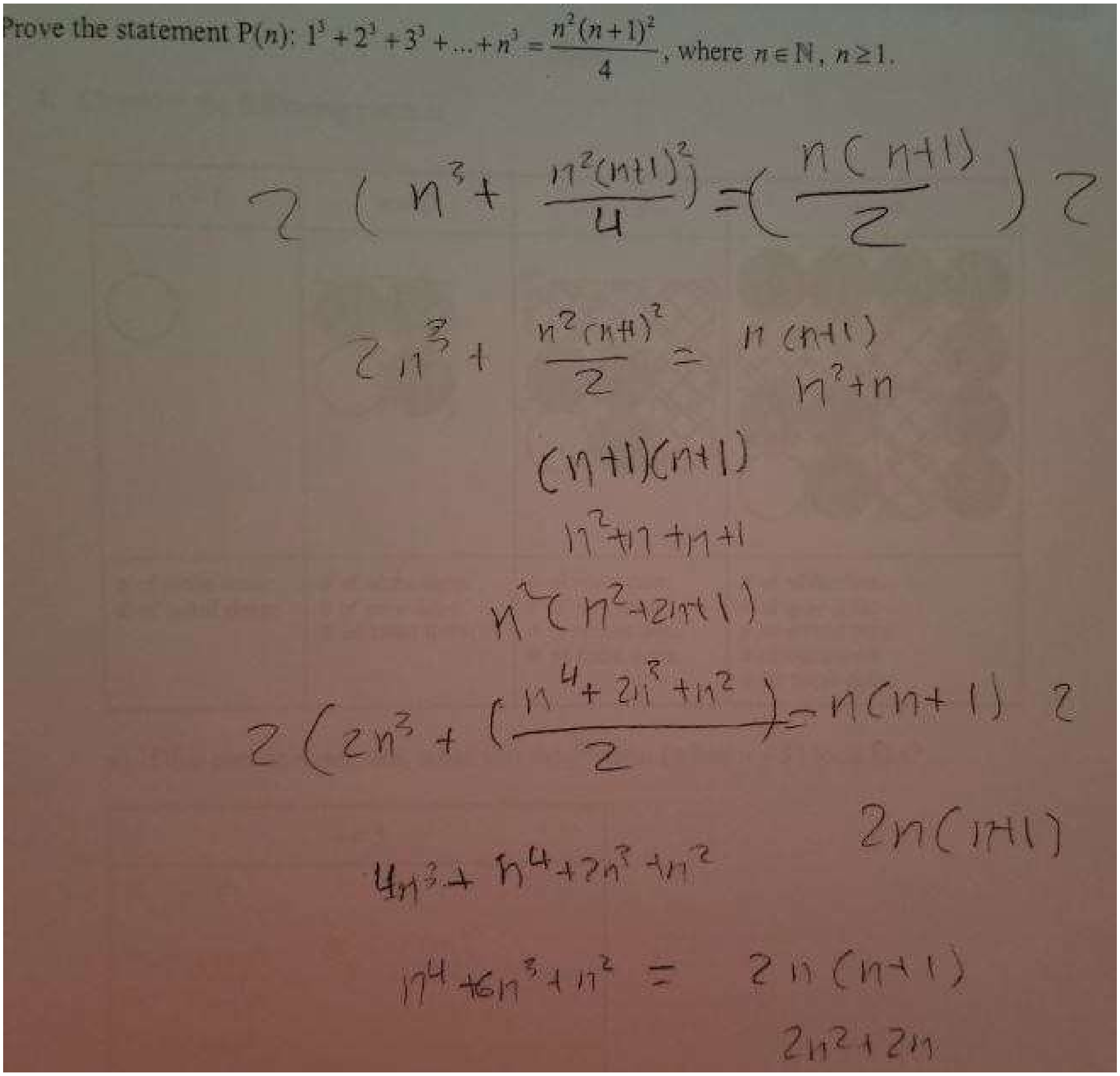


Figure 19. Example of difficulty with substituting $P(k)$ into $P(k + 1)$ and showing $P(k) \rightarrow P(k + 1)$.

Figure 19 shows that the student choose the last term from the sum of the left side of the identity and the right side of the identity from $P(n)$, and wrote an algebraic identity that did not hold, although being able to manipulate algebraically correctly the expansion of $(n + 1)^2$ or the identity $\frac{n^2(n+1)^2}{4} = \frac{(n(n+1))^2}{2^2}$.

The following is an excerpt from the interview:

> Student: So, I know the initial formula is n, n plus 1 over 2, and so I know I have to compare that. […] And then this is equal to that. And then, well, I want to move this over, so I just did it algebraically. […] and then from here, I can just like distribute this. And then here, distribute again. Oh, squared. […] I messed up on my algebra.
>
> Instructor: Mmmmm… okay. So, then, I tell you, you didn't mess up with your algebra. Your algebra is good. Then what else did you think?
>
> Student: I messed up on the formula?

The interview excerpt shows that the student notices that he or she "messed up the formula," in this case substituting $P(k)$ into $P(k + 1)$ and showing $P(k) \rightarrow P(k + 1)$.

Student difficulty with not stating the domain for the introduced variable k.

There were seven instances (88%) where students did not state the domain for the introduced variable k. Figure 20 is an example of a student who did not state the domain for the introduced variable k. The student was given the task:

Prove the statement $P(n)$: $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n - 1)n = \dfrac{n(n - 1)(n + 1)}{3}$.

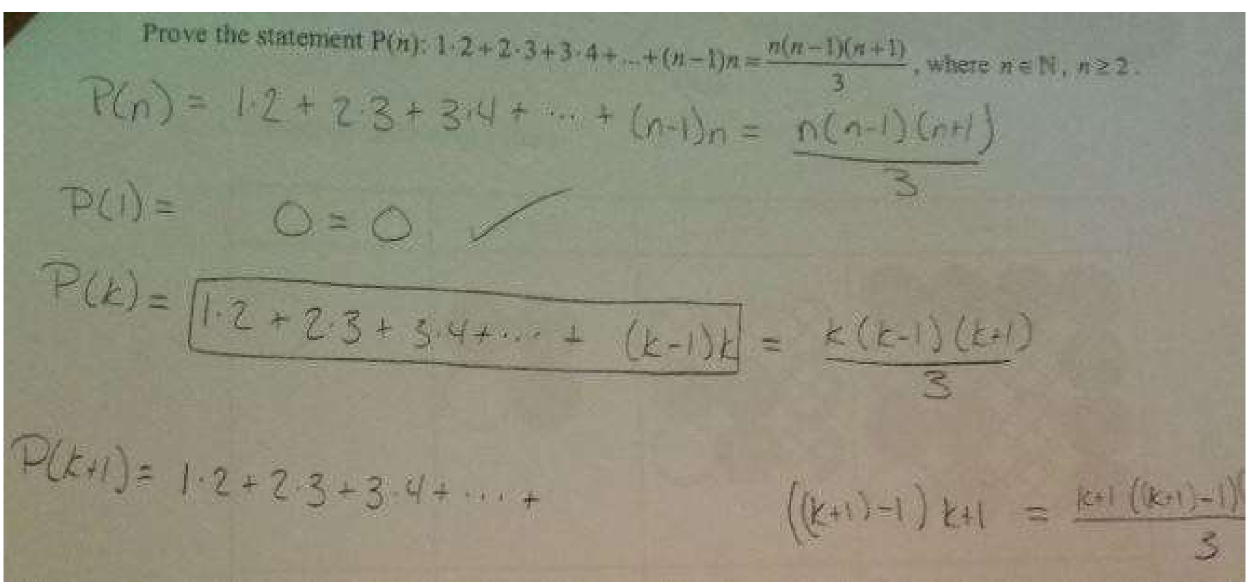


Figure 20. Example of answer missing the domain for the introduced variable k in the interview.

Figure 20 shows that the student did not state the domain for the introduced variable k during his or her interview.

Student difficulty with not stating P(k) → P(k + 1), assuming that P(k) is true.

There were four instances (50%) where students did not explicitly write $P(k) \rightarrow P(k + 1)$ assuming that $P(k)$ was true. Figure 21 is an example of a student that did not write $P(k) \rightarrow P(k + 1)$ assuming that $P(k)$ was true.

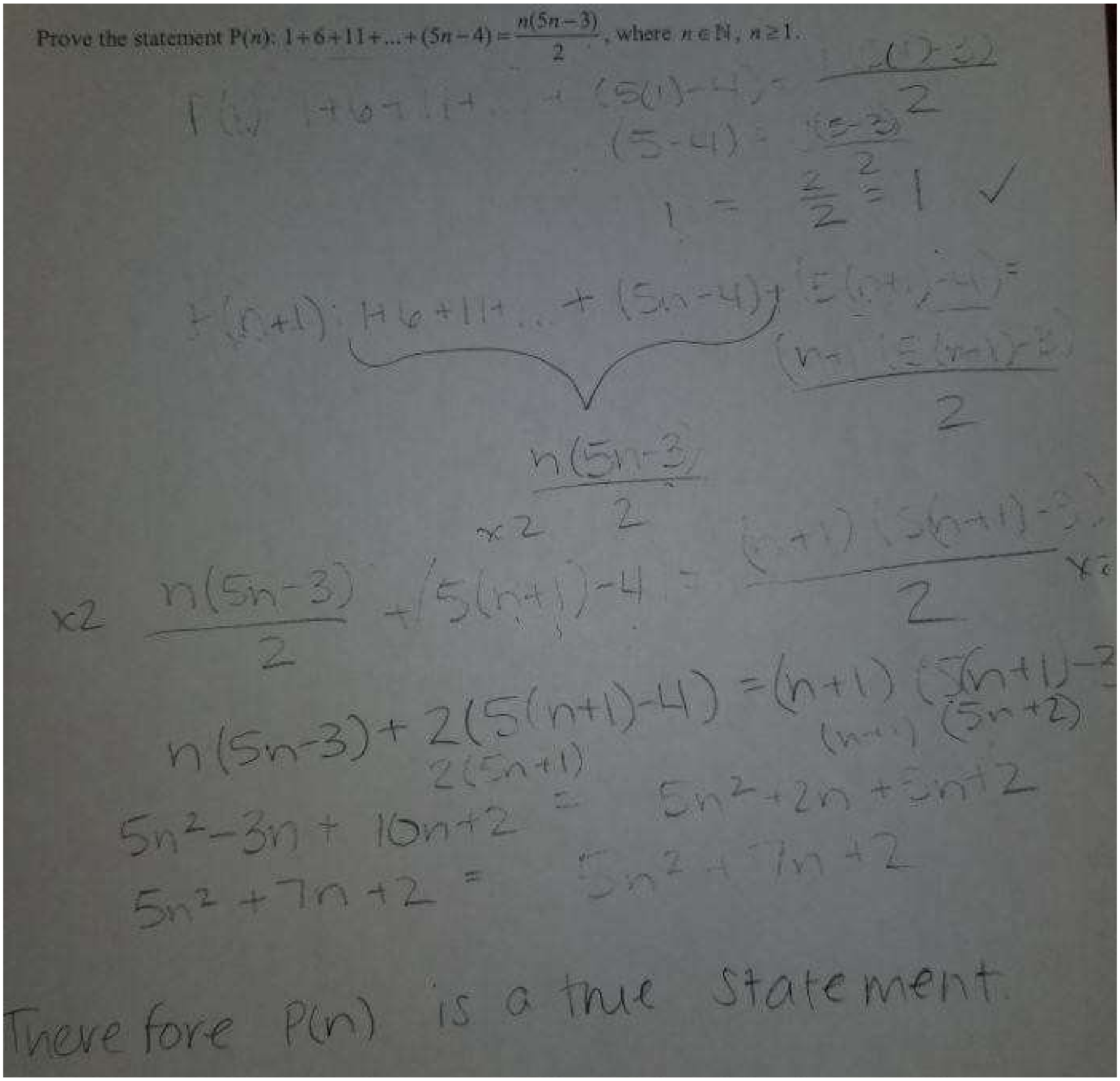


Figure 21. Example of difficulty with not stating $P(k) \rightarrow P(k + 1)$, assuming that $P(k)$ is true.

Figure 21 shows that the students completed the proof by mathematical induction without writing $P(k) \rightarrow P(k + 1)$ assuming that $P(k)$ was true

Student difficulty with invoking PMI.

There were five instances where students did not invoke PMI at the end of their proofs by mathematical induction. These difficulties include not explicitly writing "by PMI…" and invoking PMI without knowing what it represents. The task given to the student during his or her interview was:

Prove the statement $P(n)$: $1 + 6 + 11 + \ldots + (5n - 4) = \frac{n(5n-3)}{2}$, where n is a natural number, $n \geq 1$.

The following is an excerpt from the interview with the student, who completed the basis and inductive step, with observable difficulties with invoking PMI:

So, $P(k) \to P(k + 1)$ came out to be true, equal…. So, the PMI, I forgot what PMI stands for, is true for all n in the natural numbers for $n \geq 1$.

From the excerpt, the student notices the difficulty with invoking PMI when he or she does not know what it means.

Summary of analysis of students' difficulties with mathematical induction from interviews with students.

From the interviews, similar results regarding the frequency and nature of difficulties were observed by the researcher. Most of the difficulties occurred with writing the proof in the correct form with more than half of the interviews showing problems with invoking PMI, nearly all not stating the domain for the introduced variable k, and half of the interviewed students not writing that $P(k) \to P(k + 1)$ was to be shown assuming $P(k)$ to be true.

It was seen that students may invoke PMI without knowing what it means. This shows that steps were being followed to complete the proof by mathematical induction, as requested by the instructor of the course. This is also evidence of a ritualistic proof (Harel and Sowder, 1998).

Summary of Research Question 1.

Research Question 1 asked "What are the difficulties that undergraduate students have when using the method of mathematical induction, as described by Ernest's (1984) framework?"

From the analysis of Research Question 1, students experienced difficulties at each step of a proof by mathematical induction (see Table 7 and Table 8).

Task 2a, which involved polynomials, was more difficult for students compared to Task 1, which required the manipulation of exponents.

Students performed better in the basis step, with nearly every response having it in the proof by mathematical induction.

The difficulties that occurred in the inductive step were stating $P(k)$, stating $P(k + 1)$, substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$, and the algebraic manipulation of the terms to prove $P(k) \rightarrow P(k + 1)$. The most prevalent difficulty for students was correctly stating $P(k + 1)$ and the algebraic manipulations needed to prove $P(k) \rightarrow P(k + 1)$.

The most difficult for students was writing the proof in the correct form. More than half of the responses did not explicitly write the implication statement $P(k) \rightarrow P(k + 1)$ and assumed $P(k)$ to be true. Another frequent difficulty with more than a third of the responses was not stating the domain for the introduced variable k.

From the interviews with eight students, similar results were found. Nearly all of them did not state the domain for the introduced variable k. Half did not write that $P(k) \rightarrow P(k + 1)$ was to be shown assuming $P(k)$ to be true. More than half had difficulties with invoking PMI. It was also observed that students may invoke PMI without knowing what it means, simply performing steps to complete the proof.

Next, the data analysis for Research Question 2 is presented.

Research Question 2

Research Question 2 asked "How do students' self-described difficulties relate to how researchers describe students' difficulties with the method of mathematical induction?"

Inter-rater Reliability.

Twenty students' responses to Task 3 were scored by the researcher and a second rater. The inter-rater agreement analyses (Cohen, 1960) were performed using the Kappa statistic to determine the reliability between the two raters. The agreement between the two raters was 1.00 ($p < 0.01$).

Students self-described difficulties with Task 1 and Task 2a.

Table 10 follows the steps, and sub-steps, of a proof by mathematical induction. The table was used to see which parts of mathematical induction were difficult for them, based on their responses to Task 3.

Table 10

Comparison of frequencies of students' self-described difficulties and difficulties observed by researcher with Task 1 and Task 2a

Difficulties by steps and sub-steps	Frequency described by students (% out of 78)	Frequency described by researcher (% out of 156)
Difficulties in the basis step		
Omitting the basis step	0	5 (3%)
Difficulty choosing the starting point, n_0	3 (4%)	3 (2%)
Difficulty evaluating $P(n_0)$	7 (9%)	4 (3%)
Difficulties in the inductive step		
Difficulty stating $P(k)$	1 (1%)	31 (20%)
Difficulty stating $P(k + 1)$	4 (5%)	55 (35%)
Difficulty substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$	1 (1%)	19 (12%)
Difficulties with algebraic manipulations to show $P(k) \rightarrow P(k + 1)$	40 (51%)	32 (21%)
Difficulties with writing the proof in the correct form		
Not invoking PMI	7 (9%)	29 (19%)
Not stating the domain for the introduced variable k	3 (4%)	60 (38%)
Not explicitly writing that $P(k + 1)$ is to be shown assuming that $P(k)$ is true	3 (4%)	90 (58%)

It can be seen in Table 10 that the difficulties observed by researchers are far more frequent than difficulties described by students with the exception with evaluating $P(n_0)$. There were 156 responses, 78 for Task 1 and Task 2a, observed by the researcher, because a student could have difficulties with choosing the starting point, n_0, in Task 2a but not Task 1. There were

78 responses to Task 3 because there were 78 participants that could have described difficulties either with proofs by mathematical induction in general or specifically with one of the tasks.

Students' self-described difficulties with the basis step.

According to Table 10, there were three student responses that mentioned a difficulty with choosing n_0 and seven student responses that described evaluating $P(n_0)$ to be difficult compared to three responses that had difficulties with choosing n_0 and four responses that had difficulties with evaluating $P(n_0)$, as observed by the researcher.

Figure 22 is an example of a student's answer that claimed to have difficulties with choosing n_0. Figure 23 is an example of a student's answer that claimed to have difficulty with evaluating $P(n_0)$.

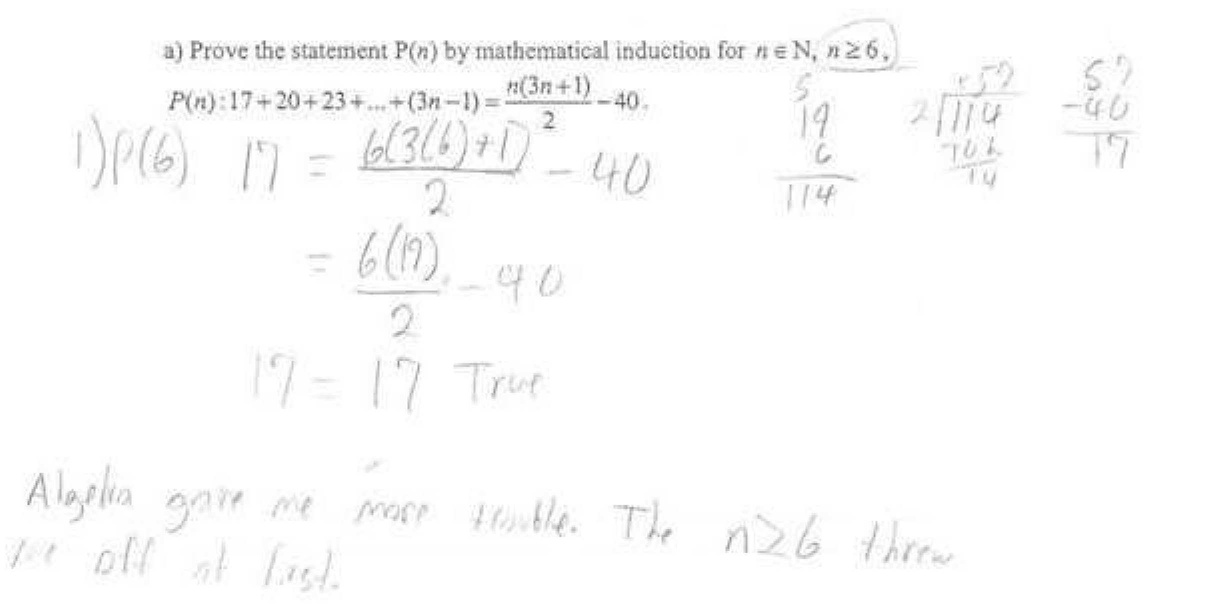


Figure 22. Example of answer with self-described difficulty choosing n_0.

Figure 22 shows that difficulties with choosing the starting point, n_0, in Task 2a were described ("The $n \geq 6$ threw me off at first."). However, it can be seen that $P(6)$ was correctly evaluated, so the researcher could not have observed this difficulty.

a) Prove the statement P(n) by mathematical induction for $n \in \mathrm{N}$, $n \geq 6$,

$$P(n): 17 + 20 + 23 + \ldots + (3n-1) = \frac{n(3n+1)}{2} - 40.$$

I basis $n = 0$

$$“17 = \frac{0(3(0)+1)}{2} - 40”$$

$$“17 = \frac{0}{2} - 40”$$

$$“17 = 0 - 40”$$

$$“17 = -40”$$

Not true

I had difficulties with the basis $n = 0$ because it was false and I didn't know how to continue.

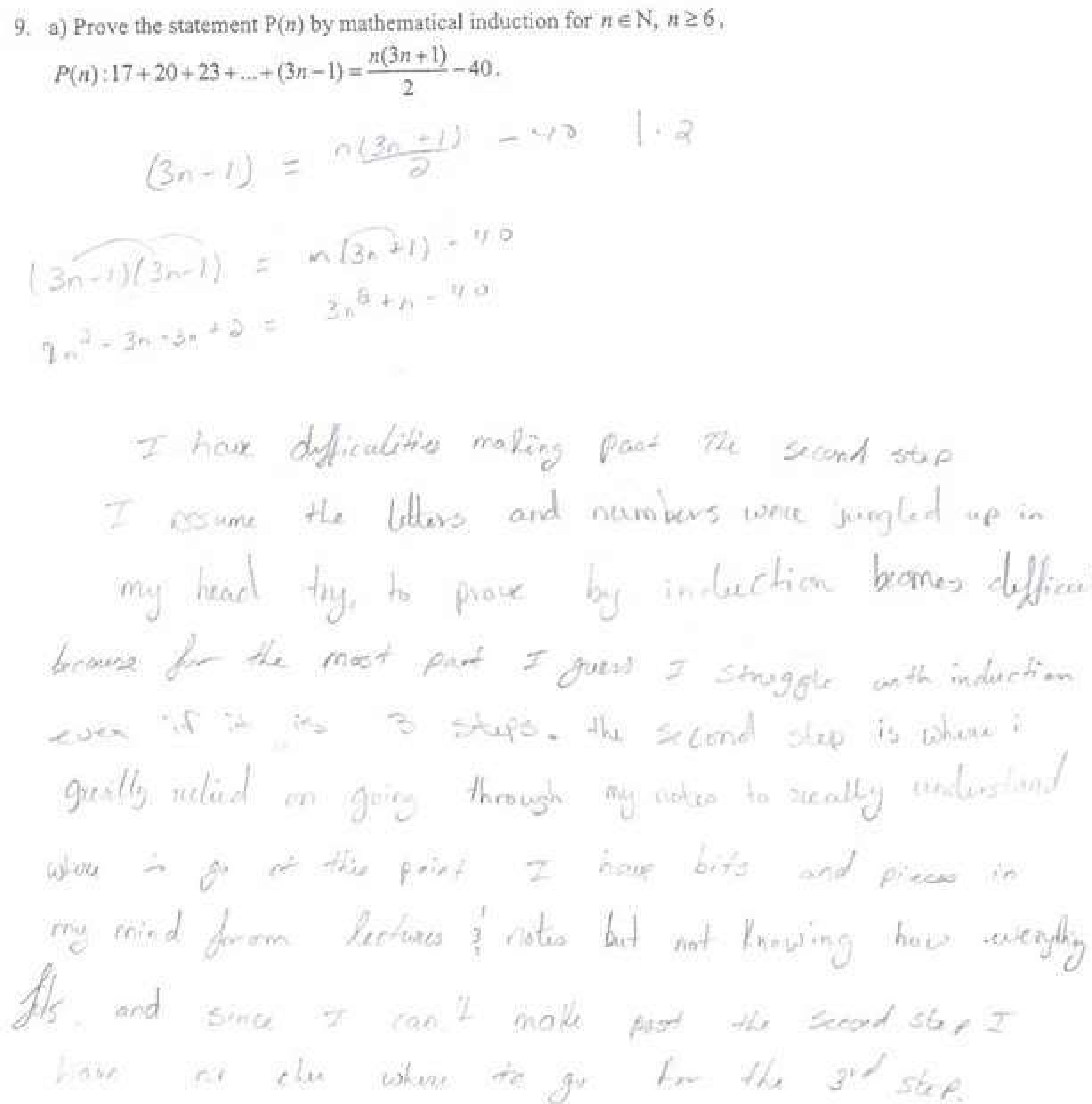
Figure 23. Example of answer with self-described difficulty with evaluating $P(n_0)$.

Figure 23 shows a description of having had difficulties with evaluating $P(n_0)$ in Task 2a: “I had difficulties with the basis $n = 0$ because it was false and I didn’t know how to continue.” The calculation continues and it is determined that the statement is not true. In comparison with student difficulties, as observed by the researcher, the difficulty stemmed from choosing the incorrect starting point, n_0, which should have been $n = 6$ and an incorrectly evaluation of the left-hand side of the identity of the statement. It is inferred that $n = 0$ was chosen since most proofs by mathematical induction begin with $n = 0$ or $n = 1$. Therefore 17 was chosen as the first term provided in the statement while substituting 0 into the sum, the right-hand side of the identity.

Students’ self-described difficulties with the inductive step.

Table 10 shows that one student response described difficulties stating $P(k)$, four student responses described difficulties writing $P(k + 1)$, one student response described difficulties with substituting $P(k)$ into $P(k + 1)$ to show $P(k) \to P(k + 1)$, and 40 responses described difficulties with the algebraic manipulations needed to prove $P(k) \to P(k + 1)$. These are compared to 31 students' responses that had difficulties stating $P(k)$, 55 responses that had difficulties writing $P(k + 1)$, 19 responses that had difficulties with substituting $P(k)$ into $P(k + 1)$ to show $P(k) \to P(k + 1)$, and 32 responses that had difficulties with the algebraic manipulations needed to prove $P(k) \to P(k + 1)$, as observed by the researcher. One can see that there is a big difference between the difficulties observed by researchers and what is described by students.

Figure 24 is an example of a student's answer that claimed to have difficulties with stating $P(k)$ and substituting $P(k)$ into $P(k + 1)$ to show $P(k) \to P(k + 1)$.



9. a) Prove the statement P(n) by mathematical induction for $n \in \mathrm{N}$, $n \geq 6$,

$$P(n): 17+20+23+\ldots+(3n-1)=\frac{n(3n+1)}{2}-40.$$

$$(3n-1)=\frac{n(3n+1)}{2}-40 \quad \Big| \cdot 2$$

$$(3n-1)(3n-1)=n(3n+1)-40$$

$$9n^2-3n-3n+2=3n^2+n-40$$

I have difficulties making past the second step I assume the letters and numbers were jungled up in my head trying to prove by induction becomes difficult because for the most part I guess I struggle with induction even if it is 3 steps. the second step is where i greatly relied on going through my notes to really understand where i go at this point I have bits and pieces in my mind from lectures & notes but not knowing how everything fits and since I can't make past the second step I have no clue where to go for the 3rd step.

Figure 24. Example of student answer with self-described difficulties stating P(k) and substituting P(k) into P(k + 1) to show P(k) implies P(k + 1).

Figure 24 shows difficulties with stating the induction hypothesis and using it to show $P(k)$ implies $P(k+1)$. Assuming that the description referred to the inductive step, the first thing to do was to write the induction hypothesis, $P(k)$ (or $P(n)$ in this case since the variable did not change). It can be seen that the student attempted to write the induction hypothesis, $P(n)$, but only included the final term and the sum. The difficulties observed by the researcher were the entire inductive step and writing the proof in the correct form.

Figure 25 is an example a student's response describing difficulties with stating $P(k + 1)$.

8. a) Prove the statement P(n) by mathematical induction for $n \in \mathrm{N}$, $n \geq 6$,

$P(n): 17 + 20 + 23 + ... + (3n - 1) = \frac{n(3n+1)}{2} - 40$.

$n = k$

$$17 + 20 + 23 + \ldots + (3k-1) = \frac{k(3k+1)}{2} - 40$$

$k \to k+1$

$$\frac{k(3k+1)}{2} - 40 + 3(k+1) - 1 = \frac{k+1(3(k+1)+1)}{2} - 40$$

$3k + 3 - 1$

$$2\left(\frac{k(3k+1)}{2} + 3k+2\right) = 2\left(\frac{k+1(3k+4)}{2}\right)$$

$$k(3k+1) + 6k + 4 = k+1(3k+4)$$

$$3k^2 + k + 6k + 4 \qquad 3k^2 + 4k + 3k + 4$$

$$3k^2 + 7k + 4 = 3k^2 + 7k + 4$$

THIS ONE, as you can see by the eraser marks, was difficult

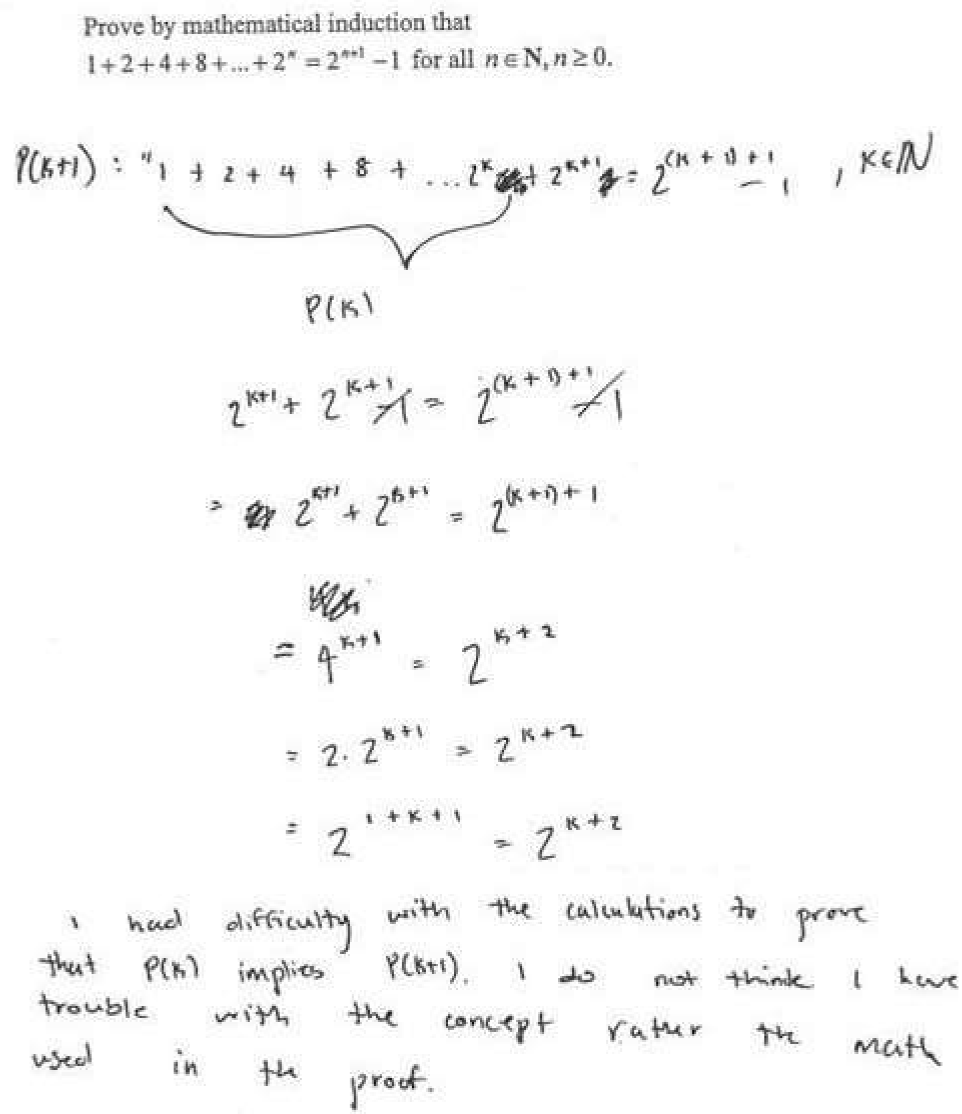
I assumed $k \to k+1$ would make $3k-1$ just $3k$ and $3k+1$ to be $3k+2$

then it was clear that it was $3(k+1)-1$ & $3(k+1)+1$ which is $3k+2$ & $3k+4$

Figure 25. Example of student's answer with self-described difficulty stating $P(k + 1)$.

Figure 25 shows that the implication $P(k) \rightarrow P(k + 1)$ was shown. However, the observed difficulties lie in writing the proof in the correct form such as explicitly stating that $P(k) \rightarrow P(k + 1)$ is to be shown and assuming that $P(k)$ is true. In the description, it is stated that the difficulties occurred with writing $P(k + 1)$ since $k + 1$ was incorrectly substituted into $P(k)$. The comment provided by the student shows that he or she performed an error, commonly done by students, by adding one to the next term of the sum, to get $3k$ from $3k - 1$, and adding one to the terms with k in the sum on the right hand side of the identity to get $3k + 2$ from $3k + 1$. This response describes this process, the mistake, and resolution of the error to complete the proof by mathematical induction.

Figure 26 is an example of a student response describing difficulties with the algebraic manipulations needed to show $P(k) \rightarrow P(k + 1)$.



Prove by mathematical induction that
$1+2+4+8+\ldots+2^n = 2^{n+1}-1$ for all $n \in \mathbb{N}, n \geq 0$.

$P(k+1)$: "$\underbrace{1 + 2 + 4 + 8 + \ldots 2^k}_{P(k)} + 2^{k+1} = 2^{(k+1)+1} - 1$, $k \in \mathbb{N}$

$2^{k+1} + 2^{k+1} \cancel{-1} = 2^{(k+1)+1} \cancel{-1}$

$= 2^{k+1} + 2^{k+1} = 2^{(k+1)+1}$

$= 4^{k+1} = 2^{k+2}$

$= 2 \cdot 2^{k+1} = 2^{k+2}$

$= 2^{1+k+1} = 2^{k+2}$

I had difficulty with the calculations to prove that P(k) implies P(k+1). I do not think I have trouble with the concept rather the math used in the proof.

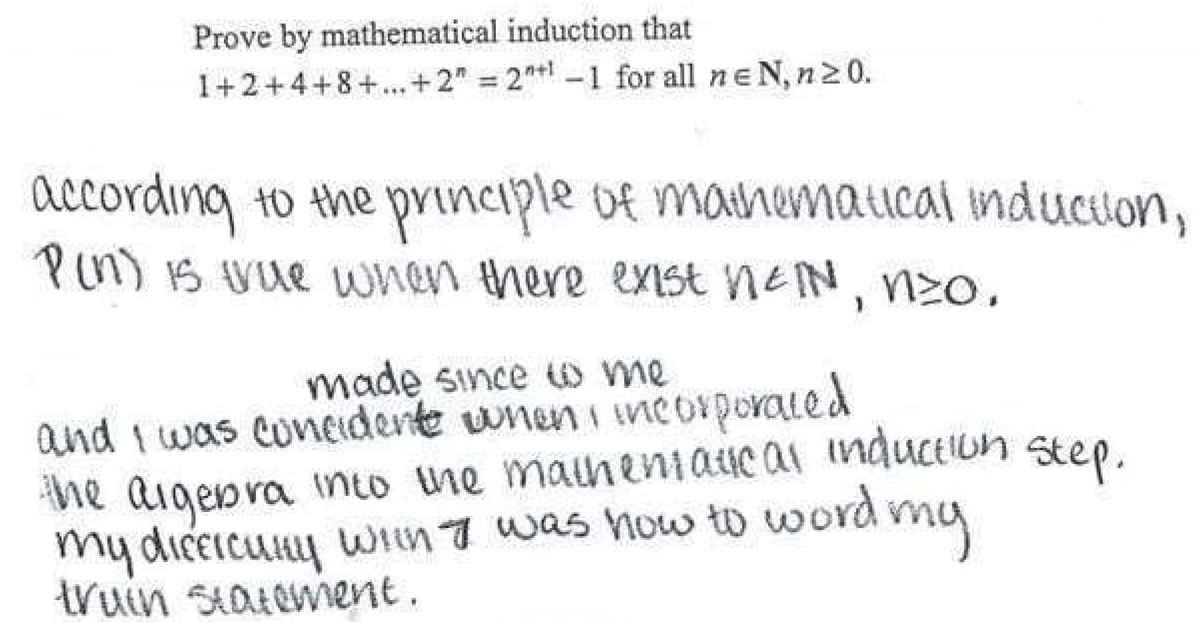
Figure 26. Example of student's answer with self-described difficulties with the algebraic manipulations needed to show $P(k) \rightarrow P(k+1)$ in Task 1.

Figure 26 shows that difficulties "with the calculations" in Task 1 in showing $P(k) \rightarrow P(k+1)$ were experienced. It can be seen that "I do not think I [had] trouble with the concept rather the math used in the proof" was written. Based on the work shown, it can also be observed by the researcher that difficulties occurred since $2^{k+1} + 2^{k+1} = 2^{(k+1)+1}$ and then $4^{k+1} = 2^{k+2}$ was written. However, $4^{k+1} \neq 2^{k+2}$. In this case, this difficulty was overcome to complete the proof by mathematical induction and the difficulties observed by the researcher and the difficulty described by the student matched.

Students' self-described difficulties with writing the proof in the correct form.

As seen from Table 10, there were two responses describing difficulties with writing the proof in the correct form. Compared to the observations by the researcher, there were 29 responses. Figure 27, Figure 28, and Figure 29 serve as examples of students' answers for each category of difficulty with writing the proof in correct form.

Figure 27 is an example of a student who claimed to have difficulties with writing the proof in the correct form by not invoking PMI.


Prove by mathematical induction that
$1+2+4+8+\ldots+2^n = 2^{n+1}-1$ for all $n \in \mathbf{N}, n \geq 0$.

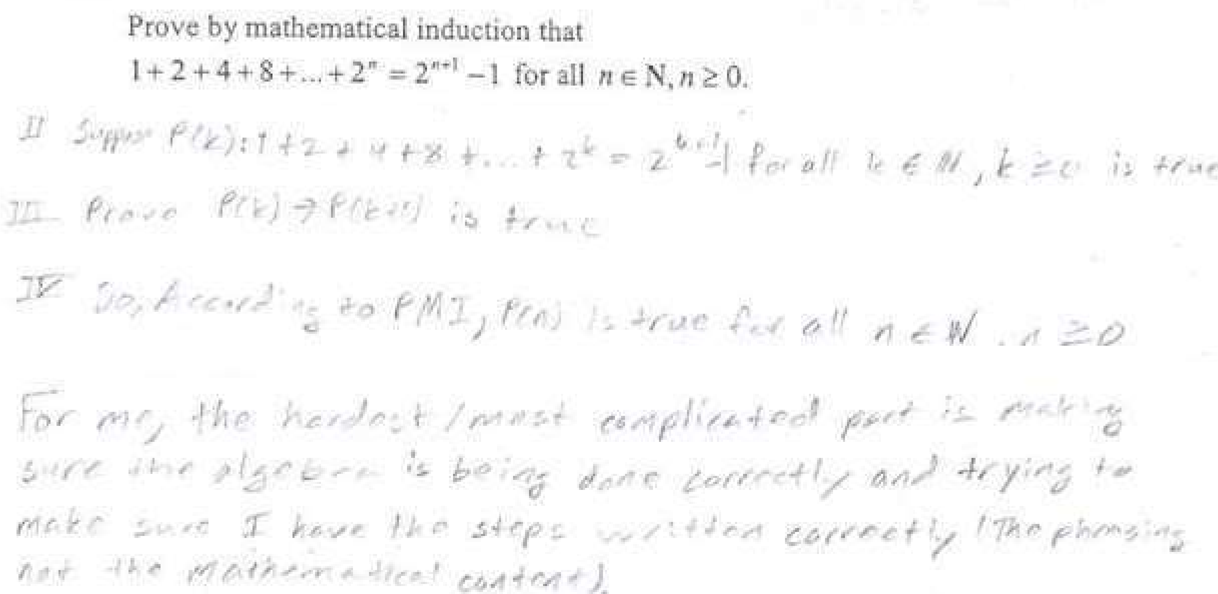
according to the principle of mathematical induction,
P(n) is true when there exist n∈N, n≥0.

made since to me
and I was confidente when I incorporated
the algebra into the mathematical induction step.
my difficulty with I was how to word my
truth statement.

Figure 27. Example of student's answer with self-described difficulty with writing the proof in the correct form by not invoking PMI.

Figure 27 shows that the proof by mathematical induction was completed to show that $P(n)$ was true for Task 1. In addition to this, the description has that he or she was uncertain if the "truth statement" was correct, in this case, it is inferred that the invocation of PMI was referenced. This difficulty is not observed by the researcher. For the observed difficulties, the student's answer states $P(k)$ for all k natural numbers greater than or equal to 0 instead of for any $n \geq 0$.

Figure 28 is an example of a student response that described difficulties with writing the proof in the correct form by not stating the domain for k and not writing $P(k) \rightarrow P(k+1)$ assuming that $P(k)$ was true.



Prove by mathematical induction that

$1+2+4+8+\ldots+2^n = 2^{n+1}-1$ for all $n \in \mathrm{N}, n \geq 0$.

II Suppose $P(k): 1+2+4+8+\ldots+2^k = 2^{k+1}-1$ for all $k \in N$, $k \geq 0$ is true

III Prove $P(k) \rightarrow P(k+1)$ is true

IV So, According to PMI, $P(n)$ is true for all $n \in W$, $n \geq 0$

For me, the hardest/most complicated part is making sure the algebra is being done correctly and trying to make sure I have the steps written correctly (The phrasing not the mathematical content).

Figure 28. Example of student's answer with self-described difficulty with writing the proof in the correct form.

Figure 28 shows that the statement, $P(n)$, was proven for Task 1. It is claimed that one of the hardest things is "trying to make sure I have the steps written correctly (the phrasing not the mathematical content)." In this case, writing the proof in correct form is referenced – this includes explicitly writing $P(k) \rightarrow P(k+1)$ assuming that $P(k)$ is true, stating the domain for the introduced variable k, and invoking PMI. However, the only observable difficulty is that $P(k)$ was not assumed to be true, instead "Prove $P(k) \rightarrow P(k+1)$ is true" was written.

Next, the summary of Research Question is presented.

Summary of Research Question 2.

Research Question 2 asked "How do students' self-described difficulties relate to how researchers describe students' difficulties with the method of mathematical induction?"

From the analysis of Research Question 2, students can convey to researchers that they experience difficulties at nearly each step of a proof by mathematical induction.

In the basis step, there were 5 students' responses to Task 1 and Task 2a that omitted the basis step, but no student responded to Task 3 referencing the possibility of this omission. There were three student answers, 4% of the 78 responses to Task 3, that described difficulties with choosing the starting point n_0. Compared to the observations by the researcher, there were three responses, 2% of the 156 students' responses to Task 1 and Task 2a, exhibiting this difficulty. The students describing this difficulty were able to overcome it. There were seven student answers, 9% of the responses to Task 3, that described difficulties with evaluating $P(n_0)$. Compared to the observations by the researcher, there were seven responses, 9% of the students' responses to Task 1 and Task 2a, exhibiting this difficulty. Students describing this difficulty could often overcome this difficulty; Figure 23 shows a students' response that could not overcome this difficulty.

In the inductive step, the students described most of their difficulties. There was one response, 1% of the students' responses to Task 3, that described a difficulty with stating $P(k)$. Compared to the observations by the researcher, there were 31 responses, 20% of the responses to Task 1 and Task 2a, that exhibited this difficulty. There were four responses, 5% of the students' responses to Task 3, that described difficulties with stating $P(k + 1)$. Compared to the observations by the researcher, there were 55 responses, 35% of the students' responses to Task 1 and Task 2a, exhibiting this difficulty. There could often overcome this difficulty; Figure 25 is an example of a student's response that could overcome this difficulty. There was one response, 1% of the students' responses to Task 3, that described a difficulty with substituting $P(k)$ into $P(k + 1)$ to show $P(k) \rightarrow P(k + 1)$. There were 19 responses, 12% of the students' responses to Task 1 and Task 2a, that were observed by the researcher having this difficulty. The most frequent difficulty described by students was the algebraic manipulations needed to show $P(k) \rightarrow P(k + 1)$ with 40 responses, 51% of the responses to Task 3. Compared to the observations by the researcher, this was not the most frequent with just 32 responses, 21% of the students' responses to Task 1 and Task 2a, exhibiting this difficulty. Students citing this difficulty could often overcome it, primarily referencing this step as the only difficulty with their proofs.

With writing the proof in the correct form, there were seven responses, 9% of the students' responses to Task 3, that cited the invocation of PMI as a difficulty. Observations by the researcher showed that there were 29 responses, 19% of the students' responses to Task 1 and Task 2a, exhibiting this difficulty. Students' responses commenting on this difficulty were split in overcoming it – students either commented on their proofs lacking the invocation of PMI and did not write it, or they were concerned whether they stated it correctly, in this case, the students did invoke PMI. For not stating the domain for the introduced variable k, there were three responses, 4% of the responses to Task 3, that cited this difficulty. Compared to the observations by the researcher, there were 60 responses, 38% of the students' responses to Task 1 and Task 2a, that exhibited this same difficulty. Students often could not overcome this difficulty. For not explicitly writing that $P(k + 1)$ is to be shown assuming that $P(k)$ was true, there were three responses, 4% of the responses to Task 3. The researcher observed that there were 90 responses, 58% of the students' responses to Task 1 and Task 2a, exhibiting this difficulty – the most frequent difficulty. Students citing this difficulty could often not overcome it.

Next, the data analysis for Research Question 3 is presented.

Research Question 3

Research Question 3 asked "What are students' convictions in mathematical induction as a method of proof?"

Students' written answers to Tasks 2a and 2b were analyzed using WMRe framework described in Chapter 1. Inter-rated agreement analyses were performed.

Inter-rater Agreement.

Twenty students' responses to Task 2b and Task 2c were scored by the researcher and a second rater. For both Task 2b and Task 2c, the initial agreement between the two raters, Cohen's (1960) kappa was 0.77 ($p < 0.01$) with CI (0.85, 0.69). The differences in coding was caused by considering some responses as Empirical Disbelief when they were displaying Unconvinced Disbelief. After re-training, the second agreement between the two raters had a Cohen's (1960) kappa of 1.00 ($p < 0.01$).

Students' Convictions.

The students' responses were coded as either Proof Conviction, Absolute Conviction, Unconvinced Disbelief, or Empirical Disbelief (see Figure 2 in Chapter 1). Students' responses that either only performed the basis step, which is only one instance of empirical evidence hence not a proof, claimed the statement was false, or made no attempt to prove Task 2a could not be coded since the categories previously mentioned regard the proof produced by the student. In addition to this, student responses who did not respond to Task 2b and Task 2c could not be coded.

One should recall Task 2:

a) Prove the statement $P(n)$ by mathematical induction for $n \in \mathbb{N}$, $n \geq 6$,

$$P(n): 17 + 20 + 23 + \ldots + (3n - 1) = \frac{n(3n + 1)}{2} - 40.$$

b) Is the statement true for $n = 10$? Justify your answer.

c) Is the statement true for $n = 5$? Justify your answer.

Table 11 shows that students' answers were placed in one of the categories of the WMRe framework described in Chapter 1. Absolute Conviction was omitted from Task 2b because the task asks about a value within the domain of the proven statement. Unconvinced Disbelief was omitted from Task 2c because the task asks about a value outside the domain of the proven statement. It should be noted that 19 students did not provide a proof of the statement, $P(n)$, in Task 2a or did not answer Tasks 2b or 2c. These 19 students were removed from the analysis for Research Question 3.

Table 11

Students' convictions and disbeliefs in Task 2a for those who produced proofs and answered Task 2b and Task 2c

	Conviction		Disbelief		
	Proof	Absolute	Unconvinced	Empirical	Total[a]
Task 2b	24 (41%)	N/A	13 (22%)	22 (37%)	59
Task 2c	34 (58%)	21 (36%)	N/A	4 (7%)	59

[a] Nineteen students did not provide enough work to prove Task 2a or did not answer Tasks 2b and 2c

Table 11 shows that 24 of the students who produced proofs of $P(n)$ from Task 2a and answered Tasks 2b and 2c did not use the proof for establishing conviction for the case when n = 10. The table also shows that more than half of the student who produced proofs of $P(n)$ from Task 2a and answered Tasks 2b and 2c used empirical calculations to determine whether $P(5)$

was true since it was not in the domain of the proven statement. The percentage in parentheses is out of 59 student responses.

Table 12 has the interaction between the responses of Task 2b and Task 2c using the WMRe categories. The interaction between the responses of Task 2b and Task 2c was necessary because it showed how students change the way conviction is established depending on the kind of question asked (e.g. whether $P(n)$ is true for n within or not within the proven domain of $P(n)$).

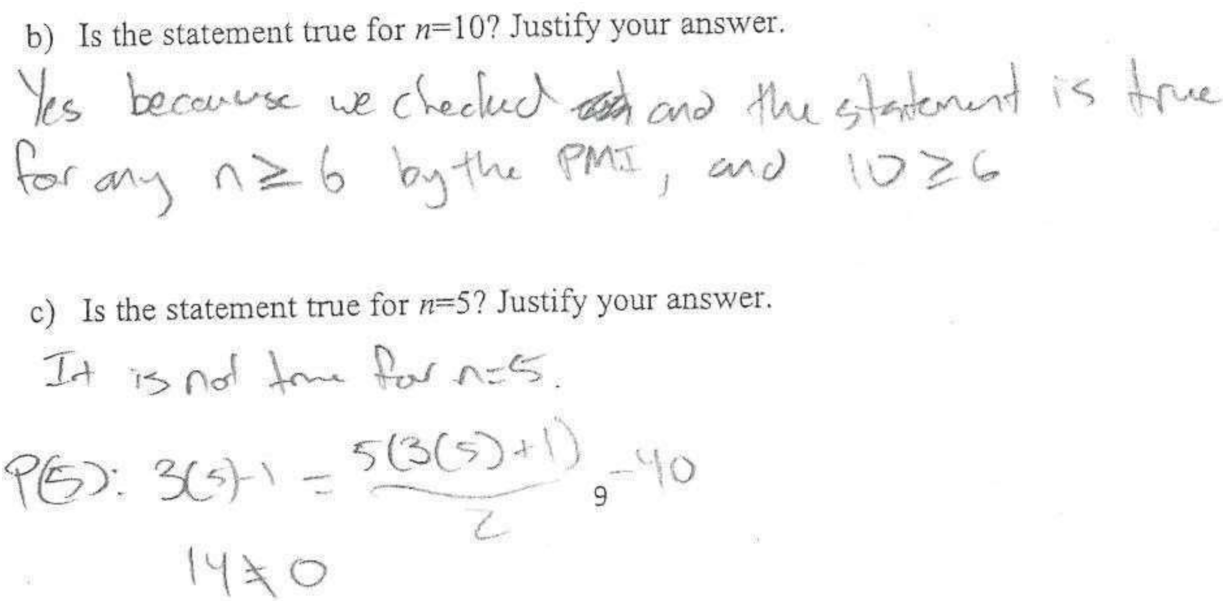
Table 12
The interactions of responses between Task 2b and Task 2c

	Task 2b result			
Task 2c result	Proof Conviction	Unconvinced Disbelief	Empirical Disbelief	Total
Proof Conviction	6 (10%)	7 (12%)	21 (36%)	34 (58%)
Absolute Conviction	17 (29%)	3 (5%)	1 (2%)	21 (36%)
Empirical Disbelief	1 (2%)	3 (5%)	0 (0%)	4 (7%)
Total	24 (41%)	13 (22%)	22 (37%)	59

Table 12 shows the number of students' responses that were coded a certain way for both Task 2b and Task 2c. The common categories were Task 2b Empirical Disbelief and Task 2c Proof Conviction as well as Task 2b Proof Conviction and Task 2c Absolute Conviction. The number in parentheses is the percentage out of 59 student responses. Each entry in the table has been analyzed by providing an example, except Task 2b Empirical Disbelief and Task 2c Empirical Disbelief which had 0 occurrences.

Task 2b Proof Conviction and Task 2c Proof Conviction.

From Table 12, there were six student responses, 10% of the responses, that displayed Proof Conviction for Task 2b and Task 2c. This means that in these responses, the proofs by mathematical induction were used for conviction in the truth of $P(10)$ and an empirical calculation was provided for $P(5)$. This combination of responses is ideally where a students' response should be because it uses the proof for conviction while acknowledging the limitations (e.g. the proof did not prove the statement for $n \leq 5$). These responses display clarity by using the proof when appropriate and using empirical evidence when necessary. Figure 29 is an example of a student's response showing Proof Conviction for both Task 2b and Task 2c.


b) Is the statement true for n=10? Justify your answer.

Yes because we checked and the statement is true for any $n \geq 6$ by the PMI, and $10 \geq 6$

c) Is the statement true for n=5? Justify your answer.

It is not true for $n=5$.

$P(5): 3(5)-1 = \frac{5(3(5)+1)}{2}$ 9 -40

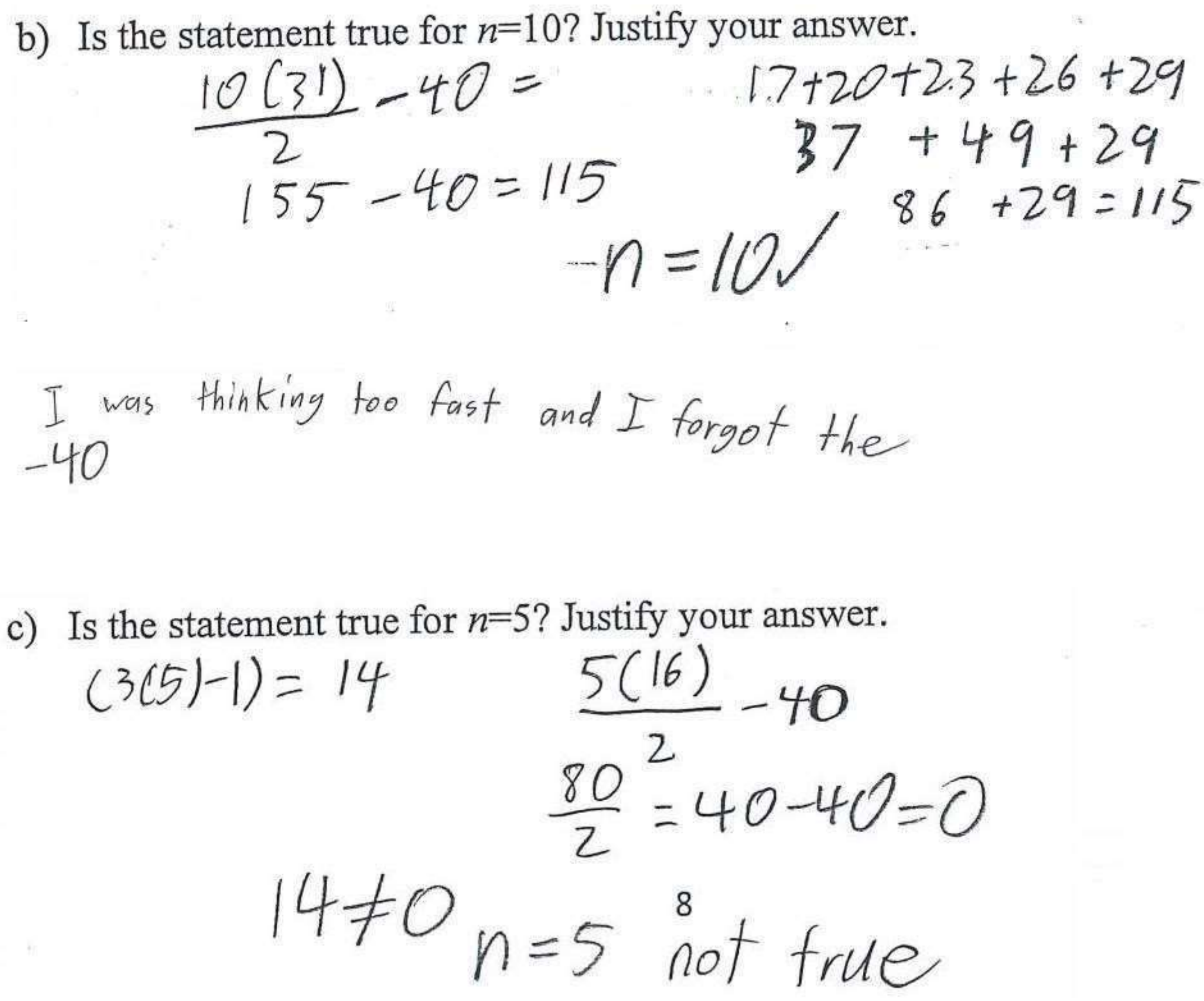
$14 \neq 0$

Figure 29: Example of student's response displaying Proof Conviction for Task 2b and Task 2c.

Figure 29 shows that the proof was used to determine whether $P(10)$ was true and $P(5)$ was justified as false by providing a calculation.

Task 2b Unconvinced Disbelief and Task 2c Proof Conviction.

As seen from Table 12, of the 59 students who produced a proof by mathematical induction for $P(n)$ in both tasks, there were seven student responses, 12% of the responses, that displayed Unconvinced Disbelief for Task 2b and Proof Conviction for Task 2c. This means that $P(10)$ was found to be true by substituting appropriate values for a calculation or a reason for $P(10)$ to be true was provided that did not include the proof while an empirical calculation for $P(5)$ was also provided. This combination of student responses was anticipated to be most frequent because students have an inclination for empirical evidence. However, this combination had a low frequency. Figure 30 is an example of a student's response showing Unconvinced Disbelief for Task 2b and Proof Conviction for Task 2c.



b) Is the statement true for n=10? Justify your answer.

$\frac{10(31)}{2} - 40 =$

$155 - 40 = 115$

$17+20+23+26+29$

$37 + 49 + 29$

$86 + 29 = 115$

$n = 10$ ✓

I was thinking too fast and I forgot the -40

c) Is the statement true for n=5? Justify your answer.

$(3(5)-1) = 14$

$\frac{5(16)}{2} - 40$

$\frac{80}{2} = 40 - 40 = 0$

$14 \neq 0$ $n=5$ not true

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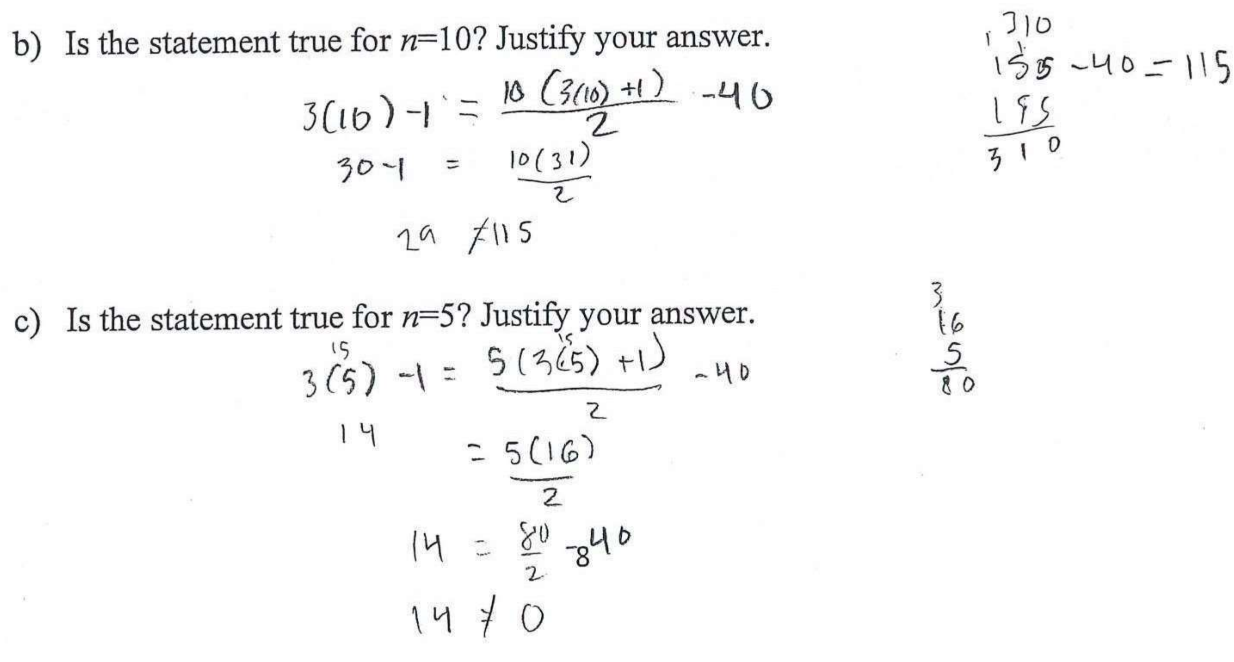
Figure 30: Example of student's response displaying Unconvinced Disbelief for Task 2b and Proof Conviction for Task 2c.

Figure 30 shows that, for Task 2b, the 5 necessary terms were added to find the sum and the formula was used for the sum to conclude equality, 115 was found for the left- and right-hand sides of the equality. Additional evidence is provided, from Task 3, by writing "I was thinking too fast and I forgot the -40" when asked about the difficulties with Task 2b – this shows that the reliance was on the empirical calculation rather than the produced proof. For Task 2c, 5 was substituted into the last term to find that it was equal to 14 while 5 was substituted into the formula for the sum to get 0, it was concluded that 14 does not equal 0, so $P(5)$ was not true.

Task 2b Empirical Disbelief and Task 2c Proof Conviction.

It can be seen from Table 12, that there were 21 student responses that displayed Empirical Disbelief in Task 2b and Proof Conviction in Task 2c. This means that these student responses found $P(10)$ to be false by errors in calculations after having produced a proof by mathematical induction and provided an empirical calculation for $P(5)$. This combination of responses is consistent because the source of conviction is empirical evidence. Compared to other combinations, this occurrence was the most frequent with 36% of the responses.

Figure 31 is an example of a student's response that displays Empirical Disbelief for Task 2b and Proof Conviction for Task 2c.


b) Is the statement true for n=10? Justify your answer.

$3(10)-1 = \frac{10(3(10)+1)}{2} - 40$

$30-1 = \frac{10(31)}{2}$

$29 \neq 115$

$155 - 40 = 115$

$\begin{array}{r} 155 \\ \underline{155} \\ 310 \end{array}$

c) Is the statement true for n=5? Justify your answer.

$3(5)-1 = \frac{5(3(5)+1)}{2} - 40$

$14 \quad = \frac{5(16)}{2}$

$14 = \frac{80}{2} - 40$

$14 \neq 0$

$\begin{array}{r} 16 \\ \underline{5} \\ 80 \end{array}$

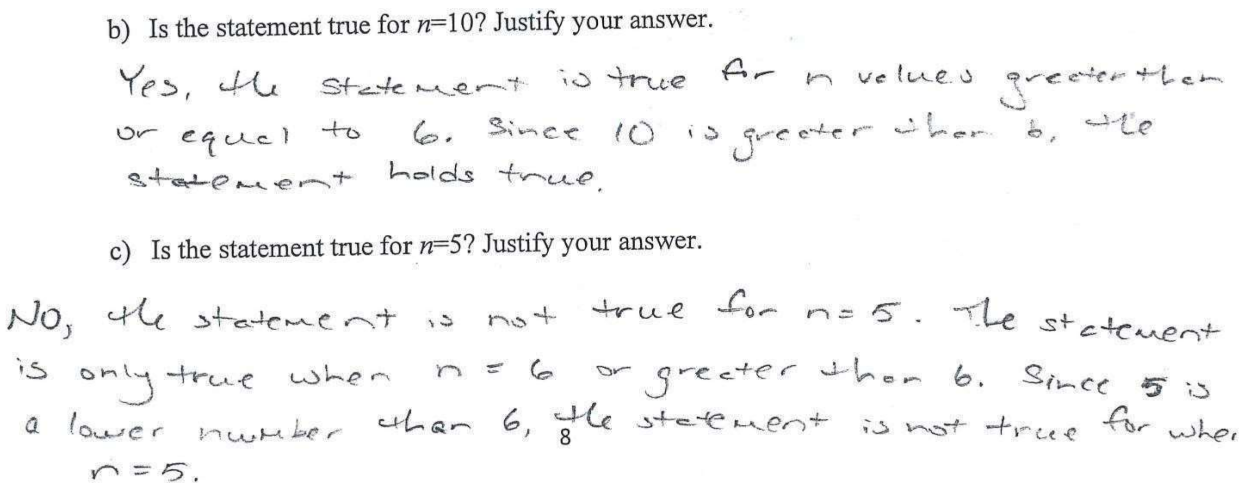
Figure 31: Example of student's response displaying Empirical Disbelief for Task 2b and Proof Conviction for Task 2c.

Figure 31 shows that only the last term was used in the summation for the addition of terms while 10 was substituted into the formula for the sum to get that 29 ≠ 115 – it is inferred that $P(10)$ was determined to be not true after $P(n)$ was proven for all integers greater than or equal to 6. For Task 2c, 5 was substituted into the last term to get 14 while 5 was substituted into the formula for the sum to get 0 – it is inferred that $P(5)$ was concluded to be not true since 14 ≠ 0 was written.

Task 2b Proof Conviction and Task 2c Absolute Conviction.

As seen from Table 12, there were 17 student responses, 29% of the responses, that displayed Proof Conviction for Task 2b and Absolute Conviction for Task 2c. This means that these student responses used their proofs by mathematical induction for conviction in the truth of $P(10)$ and stated that their proofs did not include 5 in the domain for which the statement, $P(n)$, was proven, so $P(5)$ was not true. This combination of responses is consistent because there is an over-reliance on the produced proof because the proof is the only source of conviction (e.g. any $n < 6$ is automatically false without verification). These responses were the second most frequent with 29% of the responses.

Figure 32 is an example of a student's response that displayed Proof Conviction for Task 2b and Absolute Conviction for Task 2c.


b) Is the statement true for n=10? Justify your answer.

Yes, the statement is true for n values greater than or equal to 6. Since 10 is greater than 6, the statement holds true.

c) Is the statement true for n=5? Justify your answer.

No, the statement is not true for n = 5. The statement is only true when n = 6 or greater than 6. Since 5 is a lower number than 6, the statement is not true for when n = 5.

Figure 32: Example of student's response displaying Proof Conviction for Task 2b and Absolute Conviction for Task 2c.

Figure 32 shows that the student's response used the fact that the proof showed it holds for all values of n such that $n \geq 6$, so $P(10)$ must be true. At the same time, the description showed that the statement, $P(n)$, is only true for $n \geq 6$; since $5 < 6$, $P(5)$ must be false.

Task 2b Unconvinced Disbelief and Task 2c Absolute Conviction.

As seen from Table 12, there were three instances, 5% of the responses, where a student's response displayed Unconvinced Disbelief for Task 2b and Absolute Conviction for Task 2c. This means that the student response provided a reason for why $P(10)$ was true that did not include the proof and stated that their proofs did not include 5 in the domain for which the statement, $P(n)$, was proven, so $P(5)$ was not true. This combination of responses is discordant because there is a lack of conviction in the produced proof of $P(n)$ with its domain alongside the over-reliance in the proof of $P(n)$ outside of its domain. The source of conviction shifted in these cases.

Figure 33 is an example of a student's response that showed Unconvinced Disbelief for Task 2b and Absolute Conviction for Task 2c.

b) Is the statement true for n=10? Justify your answer.

$3(10)^2 + 7(10) + 4 = 3(10)^2 + 7(10) + 4$

yes it is true

The only difficulties I found with a was the algebra. Since it got confusing since I did not have a lot of space to write.

c) Is the statement true for n=5? Justify your answer.

It is false because n is true for any natural number greater than or equal too 6.

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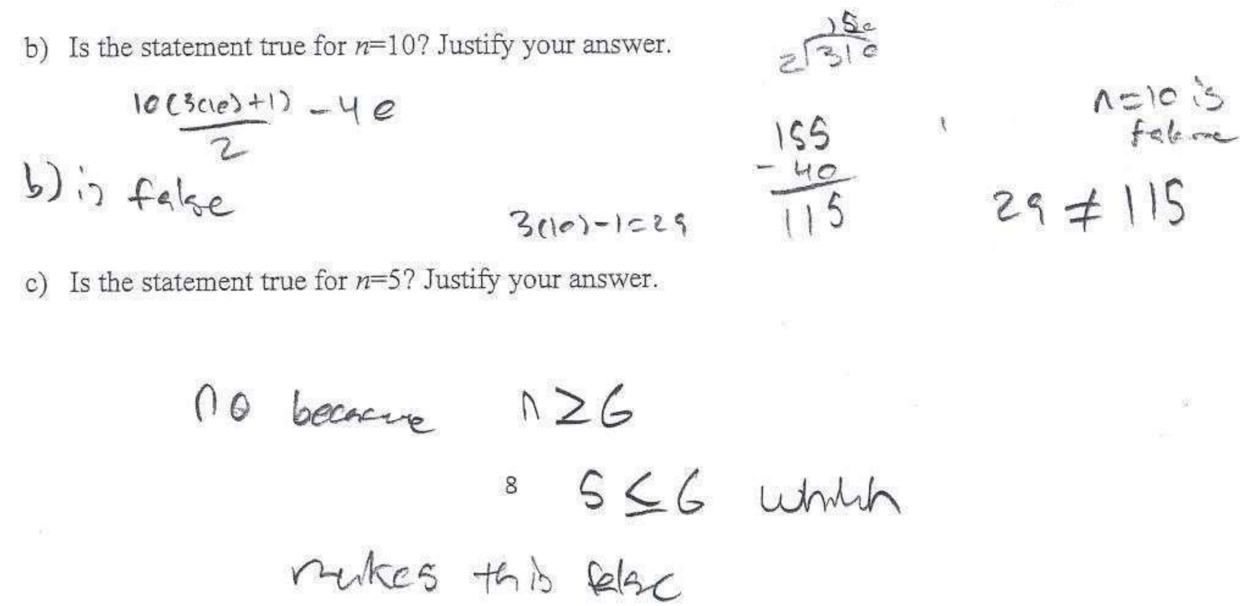
Figure 33. Example student's response displaying Unconvinced Disbelief for Task 2b and Absolute Conviction for Task 2c.

Figure 33 shows that the student's response for Task 2b relied on empirical calculations to determine whether $P(10)$ was true. The response provided additional evidence that the proof was not used in determining the truth of $P(10)$ by stating that only algebraic manipulations were difficult. For Task 2c, it was stated that "n," although the student meant the statement, is only true for $n \geq 6$.

Task 2b Empirical Disbelief and Task 2c Absolute Conviction.

As seen from Table 12, there was one instance, 2% of the responses, where a student's response displayed Empirical Disbelief for Task 2b and Absolute Conviction for Task 2c. This means that the student response provided a calculation that showed $P(10)$ to be false and stated that the proof did not include 5 in the domain for which the statement, $P(n)$, was proven, so $P(5)$ was not true. This combination of responses is also discordant because the response is relying on empirical calculations while over-relying on the proof. The source of conviction shifted.

Figure 34 is an example of a student's response that showed Empirical Disbelief for Task 2b and Absolute Conviction for Task 2c.



b) Is the statement true for n=10? Justify your answer.

$\frac{10(3(10)+1)}{2} - 40$

b) is false

$3(10)-1=29$

155
− 40
115

n=10 is false

$29 \neq 115$

c) Is the statement true for n=5? Justify your answer.

no because $n \geq 6$

$5 \leq 6$ which

makes this false

Figure 34. Example of student's response displaying Empirical Disbelief for Task 2b and Absolute Conviction for Task 2c.

Figure 34 shows that the response relied on a calculation to determine *P*(10) was false despite having just proven it. The student's response also stated that *P*(5) was false because 5 ≤ 6, so *P*(5) is false.

Task 2b Proof Conviction and Task 2c Empirical Disbelief.

As seen from Table 12, there was 1 instance, 2% of the responses, where students' responses displayed Proof Conviction for Task 2b and Empirical Disbelief for Task 2c. This means that the student response relied on the proof to determine whether *P*(10) was true and provided a calculation that contradicted the statement or proof to determine that *P*(5) was true. This combination of responses forces a student to shift the source of conviction from the proof to empirical evidence. However, this combination of responses was low. The converse of these responses, Task 2b Empirical Disbelief and Task 2c Proof Conviction, was more frequent with 21 instances (36%) – these responses relied solely on empirical evidence.

Figure 35 is an example of a student's response that displayed Proof Conviction for Task 2b and Empirical Disbelief for Task 2c.

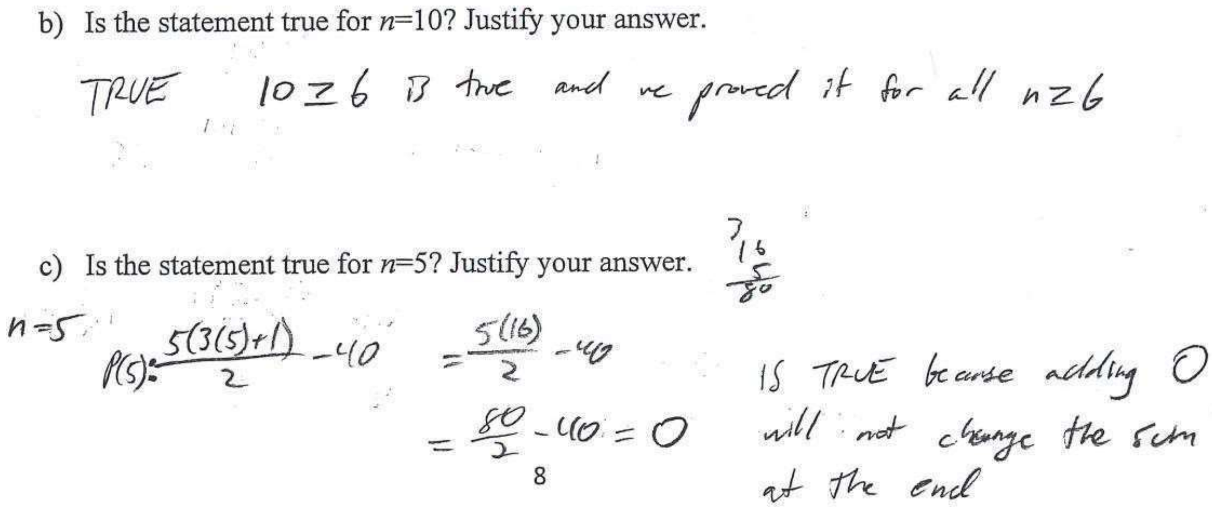


Figure 35. Example of student's response displaying Proof Conviction for Task 2b and Empirical Disbelief for Task 2c.

Figure 35 shows that the student that the student's response relied on the proof to determine whether *P*(10) was true. The response also provided a calculation that he or she believes shows *P*(5) is true, in this case, the student seemed to only concern himself or herself with the sum rather than the terms to be added.

Task 2b Unconvinced Disbelief and Task 2c Empirical Disbelief.

From Table 12, there were three instances, 5% of the responses, where students' responses displayed Unconvinced Disbelief for Task 2b and Empirical Disbelief for Task 2c. This means that the student response used means that did not include the proof to determine whether *P*(10) was true and determined *P*(5) to be true. This combination of responses is consistent since the response relies on empirical evidence for conviction. However, its frequency was low with 5% compared to other combinations that relied solely on empirical evidence, Task

2b Empirical Disbelief and Task 2c Proof Conviction (36%) and Task 2b Unconvinced Disbelief and Task 2c Proof Conviction (12%).

Figure 36 is an example of a student's response that displayed Unconvinced Disbelief for Task 2b and Empirical Disbelief for Task 2c.

b) Is the statement true for n=10? Justify your answer.

yes because when you plug in 10 for n the value is greater than zero

I feel like I had no difficulties with b

c) Is the statement true for n=5? Justify your answer.

yes it is True because when you plug in 5 for n the value is zero hence $n \in N$, $n \geq 6$

Figure 36. Example of student's response displaying Unconvinced Disbelief for Task 2b and Empirical Disbelief for Task 2c.

Figure 36 shows that the student's response provided a reason that did not include the proof when determining whether *P*(10) was true and provided additional evidence that the proof was not used for Task 2b. For Task 2c, the response cites a calculation that "the value is zero" and determines $P(5)$ to be true. It is inferred that 5 was only substituted into the sum of $P(n)$, $\frac{n(3n+1)}{2} - 40$, to get $\frac{5(3(5)+1)}{2} - 40 = 40 - 40 = 0$.

Summary of Research Question 3.

Research Question 3 asked "What are students' convictions in mathematical induction as a method of proof?"

The WMRe framework and the categories used, Proof Conviction, Absolute Conviction, Unconvinced Disbelief, and Empirical Disbelief, are defined in Chapter 1.

From Table 12, students' responses displayed a variety of answers from where they derived their convictions.

Of the 59 responses, there were six instances (10%) where students' responses displayed clarity in the produced proofs – using the proof when appropriate and using empirical evidence when necessary. These responses to Task 2b and Task 2c are ideal.

Forty-eight out of 59 student responses (81%) were consistent with the source of conviction used. The majority of students' responses were consistent in deriving conviction from empirical evidence. The frequency for these consistencies was 31 instances (53%). The frequency for using only the proof as the source of conviction was 17 out of 59 (29%). There were 4 students' responses (7%) that were discordant – their sources of conviction shifted from empirical evidence to the proof.

Summary of Chapter 4

For Research Question 1, the E-a framework was used to analyze the difficulties students faced with Task 1, Task 2a, and interviews. Students experienced difficulties at each step of a proof by mathematical induction. Overall, students performed better in the basis step with a mean score of 0.88 out of 1 for Task 1 and 0.84 out of 1 for Task 2a. Students performed the worst with writing the proof in the correct form with mean scores of 1.09 out of 2 for Task 1 and 0.92 out of 2 for Task 2a. The least frequent difficulties occurred in the basis step. These difficulties were choosing the correct initial value, n_0, and evaluating $P(n_0)$ with 2% and 3% of the students' responses to Task 1 and Task 2a exhibiting these difficulties, respectively. The most frequent difficulty occurring in the inductive step was stating $P(k + 1)$ with 35% of the 156 responses. The most frequent difficulties occurred with writing the proof in the correct form. These were stating the domain for the introduced variable k and writing that $P(k + 1)$ was to be shown assuming $P(k)$ was true with 38% and 58% of the students' responses to Task 1 and Task 2a exhibiting these difficulties, respectively. The analysis of the interviews with eight students showed a similar result. In particular, it was observed that students may invoke PMI without knowing what it represents.

For Research Question 2, the E-a framework was also used to analyze the difficulties that students described having, asked in Task 3. Students were able to describe difficulties at nearly each step of a proof by mathematical induction. The most frequent difficulty cited by 51% of the student responses were the algebraic manipulations needed to prove $P(k) \rightarrow P(k + 1)$. The least frequent references for difficulty by students came from the basis step, claiming difficulties with choosing the starting point, n_0 (4% of the responses), and claiming difficulties with evaluating $P(n_0)$ (9% of the responses) for their proofs. Student describing difficulties with the basis step and the inductive step could often overcome them. However, students describing difficulties with writing the proof in the correct form could not overcome the difficulty.

For Research Question 3, the WMRe framework was used to analyze 59 students who produced proofs by mathematical induction for Task 2a and responded to Task 2b and Task 2c. Students displayed a variety of answers from where they derived their convictions. However, 48 out of 59 responses (81%) were consistent from where conviction was derived. There 31 responses (53%) that were consistent from where they derived conviction – either their proofs or empirical evidence. There were 4 responses (7%) that were discordant meaning that from where conviction was derived shifted from empirical evidence to the proof.

In the next chapter, the analysis for each research question, the limitations of this study, and the implications for teaching are discussed. The results of this study are compared with previous research results reported in literature.

Chapter 5: Discussion and Conclusions

Introduction

In this chapter, the results of the study are discussed, together with the implications for teaching, and the limitations of the study are addressed. The results of each research question are discussed.

Research Question 1 Results and Discussion

Research Question 1 asked “What are the difficulties that undergraduate students have when using the method of mathematical induction, as described by Ernest’s (1984) framework?”

The mean scores for Task 1 and Task 2a were 7.01 and 6.69 (from 0 lowest to 10 highest), respectively. These mean scores were greater than other studies – the results from Kong (2003) showed that the mean scores were very low. However, the mathematical induction tasks used in this study were only algebraic identities (series and sums) and were closely related to introductory problems and examples found in textbooks (Epp, 2010; Rosen, 2012). The findings for Research Question 1 were confirmatory with previous research (Avital and Hansen, 1976; Avital and Libeskind, 1978; Ernest, 1984; Reid, 1992; Walter, 1972; Dubinsky and Lewin, 1986; Segal, 1998; Kahn, Anderson, Austin, Barnard, Jagger, and Chetwynd, 1998; Anderson, Austin, Barnard, and Jagger, 1998).

The mean scores for the basis steps of Task 1 and Task 2a were 0.88 and 0.84 (from 0 lowest to 1 highest), respectively. This means that the students were able to complete the basis step in nearly all cases, likely due to the basis step relying on checking an identity using calculations after substituting a value into a variable. The most frequent difficulty in the basis step was omitting the basis step with 5 out of 156 responses (3%) of the responses missing it from their proofs by mathematical induction. This result is confirmatory with other studies (Avital and Libeskind, 1978; Baker, 1996; Stylianides et al., 2007).

The inductive step caused problems for the students in various ways. This step had a mean score of 6.06 out of 7 for Task 1 and 5.79 out of 7 for Task 2a. Stating $P(k + 1)$ was difficult for students with an overall mean score of 0.84 out of 1 between Task 1 and Task 2a. This was the most frequent difficulty with 35% of the responses having it. A source of problems for the students occurred when they needed to state $P(k + 1)$ because $k + 1$ needed to be substituted into $P(k)$ which included both $3k – 1$ and $3k + 1$. Students would simply add 1 to one or both of those terms which resulted in $3k$ and $3k + 2$ respectively – this would leave it not possible to show that $P(k) \rightarrow P(k + 1)$. Brown (2003) and Harel (2001) found something similar in how students represent a successor with sequences and series. In addition to this, the algebraic manipulations of terms caused problems for students when $P(k) \rightarrow P(k + 1)$ needed to be proven. The mean scores for the algebraic manipulations needed to prove $P(k) \rightarrow P(k + 1)$ were 3.43 out of 4 for Task 1 and 3.54 for Task 2a. This cause for this was likely the required knowledge of exponents needed in Task 1. For example, in Task 1, a common error was adding the terms $2^{k+1} + 2^{k+1}$ to get 4^{k+2}. In general, stating $P(k + 1)$ and the necessary prior mathematical knowledge, such as the law of exponents, is of great importance for a successful proof by mathematical induction.

Writing the proof in the correct form was seen to be absent for many students, with the lowest mean scores of any step. With a mean score of 1.09 out of 2 for Task 1 and 0.92 for Task 2a. The students did not state the domain for the introduced variable k, with a mean score of 0.19 out of 0.50 for Task 1, 0.14 out of 0.50 for Task 2a, and 38% of the responses. 58% of the responses did not explicitly write $P(k) \rightarrow P(k + 1)$ assuming $P(k)$ to be true with a mean score of

0.42 out of 0.50 for Task 1 and 0.34 for Task 2a. 19% did not invoke PMI with a mean score of 0.48 out of 1 for Task 1 and 0.44 out of 1 for Task 2a. The students viewed writing the proof in the correct form as a formality. This may be because they were taught to write down steps, such as the basis and inductive steps, to guide their proof. Similar results were found by Stylianides, Stylianides, and Philippou (2007). It was also possible for students to invoke PMI without knowing what it meant, this was seen in an interview with a student who invoked PMI but did not know what it stood for (see Chapter 4 Summary of Research Question 1). This is an example of a student's "ritualistic" way of proving (Harel and Sowder, 1998) – the student follows steps to produce what is requested, in this case, a proof by mathematical induction.

Research Question 2 Results and Discussion

Research Question 2 asked "How do students' self-described difficulties relate to how researchers describe students' difficulties with the method of mathematical induction?"

Students' responses to Task 3 showed that they experienced difficulties at each step of a proof of by mathematical induction. There were 156 responses to Task 1 and Task 2a observed by the researcher while 78 students' responses to Task 3 describing their difficulties either with proofs by mathematical induction in general or specifically with one of the tasks.

For the basis step, students and researcher were similar in identifying difficulties, although few of them. Three out of 78 (4%) responses described difficulties with choosing the starting point n_0 while 3 out of 156 (2%) responses were observed by the researcher having this difficulty. Seven out of 78 (9%) responses described difficulties with evaluating $P(n_0)$ while 4 out of 156 (3%) responses were observed by the researcher having this difficulty. No student claimed difficulty by omitting the basis step. Similar to Dubinsky and Lewin (1986), the students who omitted the basis step and did not claim difficulties with it, did not understand the necessity of verifying the basis step therefore they could not comment on it.

Most of the students self-described difficulties occurred in the inductive step. Four out of 78 (5%) responses described difficulties with stating $P(k + 1)$ while 55 out of 156 (35%) responses were observed by the researcher having this difficulty. Forty out of 78 (51%) responses described difficulties with the algebraic manipulations needed to prove $P(k) \rightarrow P(k + 1)$ while 32 out of 156 (21%) responses were observed by the researcher having this difficulty. Students' responses showed that they were more concerned with the correctness of the manipulations in proving $P(k) \rightarrow P(k + 1)$ than the reason for why the manipulations may be incorrect – if $P(k + 1)$ is incorrect, then $P(k) \rightarrow P(k + 1)$ will not be true.

There were few responses describing difficulties with writing the proof in the correct form while there were many responses observed by the researcher having difficulties. There were three out of 78 (4%) responses describing difficulties with not stating the domain for the introduced variable k while 60 out of 156 (38%) were observed by the researcher having this difficulty. There were three out of 78 (4%) responses that described having difficulties with not writing that $P(k + 1)$ was to be shown assuming $P(k)$ to be true while 90 out of 156 (58%) responses were observed by the researcher having this difficulty. The students did not describe writing the proof in the correct form as a difficulty in most cases. The large discrepancy between the number of responses describing difficulties with the inductive step and with writing the proof in the correct form compared to the observations by the researcher shows that students are more concerned with the manipulations to prove $P(k) \rightarrow P(k + 1)$. These findings are similar to Stylianides et al. (2016) and Demiray and Boston (2017). There were seven out of 78 (9%) responses describing difficulties with invoking PMI while 29 out of 156 (19%) responses were observed by the researcher having this difficulty. Similar to Fischbein (1980), students seemed to rely on the productivity of manipulations, rather than calculations, for their proofs by mathematical induction, as seen by students who did not invoke PMI and ended their proofs after proving $P(k) \rightarrow P(k + 1)$. However, it should be noted that it is possible for students to state something besides $P(n)$ was proven, as seen by a student who claimed $P(k + 1)$ was proven by PMI (see Figure 14 in Chapter 4).

Research Question 3 Results and Discussion

Research Question 3 asked "What are students' convictions in mathematical induction as a method of proof?"

For Task 2b, the student was asked whether $P(10)$ was true or false with justification. Task 2c asked whether $P(5)$ was true or false with justification. There were 59 students who produced proofs by mathematical induction for Task 2a and answered Task 2b and Task 2c. The students' responses to Task 2b and Task 2c were used to determine the sources of conviction based on whether the natural number in question was within the domain of the proven statement, $P(n)$, in Task 2a.

Nearly all responses were consistent from where conviction was derived with 48 out of 59 responses (81%). A majority of responses, 31 out of 59 (53%), were consistent with relying on empirical evidence for conviction. Empirical evidence was used to determine whether $P(10)$ was true and provided a calculation for $P(5)$. However, a frequent occurrence, 21 of these 31 students (68%), was that students determined $P(10)$ to be false after having proven $P(n)$ for $n \geq 6$. This error was caused by using only the last term in the sum for the calculation. These 31 responses fall directly in line with Fischbein (1980) who stated that students rely on the productivity of calculations. The students were also asked about their difficulties with Task2b and Task 2c and the responses showed that they were concerned with the correctness of their calculations rather than whether the proof could be used instead. Students often rely on a single source for conviction, empirical evidence or the proof, regardless of the value in question.

Seventeen out of 59 responses (29%) were consistent with over-relying on the proof for conviction. The proof was the only source of evidence for determining $P(10)$ to be true and $P(5)$ to be false. This finding is similar to Stylianides et al. (2007). This dismissal of values not within the proven domain of the statement likely comes from the authority of the instructor.

Four out of 59 responses (7%) were discordant – their sources of conviction shifted from empirical evidence to the proof. It is possible that these students did not know how to substitute 5 into $P(n)$ to answer Task 2c, so they used the next available source, the proof, to answer it. The students who answered this way only provided evidence that the proof was not used in determining the truth of $P(10)$ – they did not comment on $P(5)$. More research is needed on these responses that are discordant.

Implications for Teaching

The results of this study have provided information for researchers and educators. The teaching and learning of proofs by mathematical induction needs support to help students perform well on tasks such as the ones presented in this study. Emphasis should be placed on the inductive step and writing the proof in the correct form as those steps of a proof by mathematical induction caused the most problems for students.

A solution that will help students with the inductive step is understanding summations as some students only tended to the last term of the sum which resulted in the inability to prove $P(k) \rightarrow P(k + 1)$.

A solution for students to find proofs by mathematical induction convincing is explaining the purpose of the inductive step. Since students often did not write their proofs in the correct form, it would be beneficial to explain the purpose of stating the domain for k, explicitly writing $P(k) \rightarrow P(k + 1)$ is to be shown while assuming $P(k)$ to be true, and invoking PMI. These explanations will help students understand what role each step, and sub-step, does in constructing a successful proof by mathematical, as described by the E-a framework. To allow students to see the limitations of their proofs and to not over-rely on what they have just proven, an example of a proof that does not encompass all values of n for which $P(n)$ is true should be provided. For example, a task that could be given to students is:

> Prove by mathematical induction the statement $n^2 < 2^n$ for all natural numbers greater than or equal to 5 (Avital and Libeskind, 1978).

It should be noted that $P(1)$ is true since $1 < 2$. The task is indeed true for all natural numbers greater than or equal to 5. However, should a student prove it and then be asked whether $P(1)$ is true or false, the student may encounter a conflict such as his or her proof not encompassing all values for which $P(n)$ is true thus allowing the student to refine how convincing a mathematical proof is and how reliant he or she should be with a proof by mathematical induction regarding the domain.

Limitations of this Study

The limitations for the study include the small number of students, tasks that only use algebraic identities, and the need for more in-depth interviews.

Summary of this Study

The research questions this study answered were:

1. What are the difficulties that undergraduate students have when using the method of mathematical induction, as described by Ernest's (1984) framework?
2. How do students' self-described difficulties relate to how researchers describe students' difficulties with the method of mathematical induction?
3. What are students' convictions in mathematical induction as a method of proof?

Following the E-a framework, described in Chapter 1, students experienced and described difficulties at each step of a proof by mathematical induction. Students performed well on the basis step because it is a calculation. They focused on the algebraic manipulations in the inductive step to prove $P(k) \rightarrow P(k + 1)$ and treated this as the conclusion of their proofs. Because of this, students often did not write the proof in the correct form and did not comment on this difficulty.

Successful proofs by mathematical induction require prior mathematical content knowledge, like exponent laws, and the understanding of definitions, such as summations. Students should be taught to understand the purpose of writing the proof in the correct form because this may help them appreciate the inductive step. This may be done by first placing emphasis on the meaning of The Principle of Mathematical Induction since proofs by mathematical induction require the invocation of PMI.

Following the WMRe framework, students often relied solely on empirical evidence or proofs for conviction. Although uncommon with 4 out of 59 responses, it was possible for students to shift their sources of conviction from empirical evidence or proofs to the other to answer questions about proven statements.

The teaching and learning of mathematical induction need adjustments that are conductive to providing students with conviction in what is proven. The understanding of The

Principle of Mathematical Induction may provide students with a reason for being convinced by what they have proven. The statement and proof should be convincing enough for students to use the proof when appropriate like values within the proven domain and empirical evidence when necessary such as values outside the domain of the proof.

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LIST OF APPENDICES

APPENDIX **PAGE**

APPENDICES

Appendix A: Interview Protocol

During the interviews, students will be asked by the instructor to describe to prove a statement that they have not seen or proven before. These proofs will be based on their assignments, tests, quizzes, exams. During the students' descriptions of their proof constructions, only comments of a prompting nature will used (e.g., "continue," or "say more"). Once a problem is solved, probing questions will be used to elicit rationales about their proof construction, or to satisfy the need for more information or clarification about the students' understanding or their difficulties. The following probing questions that may be asked to students are:

1. What is your meaning of [*name a specific mathematical concept*]?
2. What technique of proof did you use? Can you justify your choice?
3. Can you identify the hypothesis/conclusion?
4. What happens with your proof construction if we remove this part of the hypothesis…, does your prove still hold?
5. Where exactly in your proof did you use this part of the hypothesis…?
6. Can you justify how you advanced in your proof from this step… to this step…?
7. You use this statement … in your proof, can you justify it? Why does it hold?
8. What was difficult about this proof? Why?
9. Would the statement still be true for this new domain? Why?
10. Could you identify a new domain for this statement that is still true? Why does it still work?
11. Is the statement true for [*this specific/special case*]?

12. Imagine you had a friend who didn't understand [*this part of the proof*], how would you explain it to him or her?
13. How could you generalize/specialize this proof?
14. (Before proving the statement) Do you believe the statement is true?
15. How could you prove this statement a different way?

Appendix B: IRB Approval

TEXAS A&M UNIVERSITY
CORPUS CHRISTI

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Division of Research and Innovation
6300 OCEAN DRIVE, UNIT 5844
CORPUS CHRISTI, TEXAS 78412
Office 361.825.2497

Human Subjects Protection Program — Institutional Review Board

DATE: April 15, 2019

TO: Valentina Postelnicu, College of Science & Engineering

CC: Mario Gonzalez, College of Science & Engineering

FROM: Office of Research Compliance

SUBJECT: Expedited IRB Approval

On 4/15/2019, the Texas A&M University-Corpus Christi Institutional Review Board IRB reviewed the following submission by expedited review under expedited category: Expedited Category 7, Research on individual or group characteristics or behavior

Type of Review:	Initial Review
Protocol Title:	Undergraduate Students' Understanding of Proofs
Investigator:	Valentina Postelnicu
IRB ID:	143-18
Funding Source:	None
Documents Reviewed:	143-18 IRB Protocol Undergraduates Understanding of Proofs VP Apr 13 143-18 Classroom Recruitment Script VP Apr 13 143-18 Information Sheet VP Apr 13 143-18 Notes VP Apr 13 Consent Form VP Apr 13 CITI Basic MAG CITI Basic VP

The IRB has **approved** this submission from 4/15/2019 to 4/14/2020. You may now begin the research project.

Reminder of Investigator Responsibilities: As principal investigator, you must ensure:

1. **Informed Consent:** **Ensure informed consent processes is followed** and information presented ensures individuals can voluntarily decide whether or not to participate in the research project.

   Attached are approved consent forms. Use the latest IRB-approved consent forms to consent subjects.

2. **Continuing Review:** Before 4/14/2020, you are to submit a continuing review form. If continuing review approval is not granted before the expiration date, **the protocol expires, and all research activities must stop**.

3. **Amendments:** This approval applies only to the activities described in the IRB submission and does not apply should any changes be made. **Any changes require an amendment to the IRB.** The Amendment must be approved before any change is implemented.

4. **Completion Report:** **Upon completion of the research project (including data analysis and final written papers), a Completion Report must be submitted.**

5. **Reportable Events:** Reportable events must be reported to the Research Compliance Office immediately.

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TEXAS A&M UNIVERSITY
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Please do not hesitate to contact the Office of Research Compliance with any questions at irb@tamucc.edu or 361-825-2497.

Respectfully,

Rebecca Ballard, JD, MA, CIP

Digitally signed by Rebecca Ballard, JD, MA, CIP
Date: 2019.04.15 16:16:29 -05'00'

Office of Research Compliance

Appendix C: Interview tasks

Use the Principle of Mathematical Induction to prove that

$$1 + 2 + 4 + 8 + \ldots + 2^n = 2^{n+1} - 1 \text{ for all } n \in \mathbb{N}, n \geq 0.$$

Prove by mathematical induction for $n \in \mathbb{N}$, $n \geq 6$: $n^2 < 2^n$

Is the statement true for $n = 10$? Justify your answer.

Is the statement true for $n = 5$? Justify your answer.

Use the Principle of Mathematical Induction to prove that

$$1 + 3 + 9 + 27 + \ldots + 3^n = \frac{3^{n+1} - 1}{2} \text{ for all } n \geq 0.$$

Prove the statement $P(n)$: $2 + 4 + 6 + \ldots + 2n = n^2 + n$, where n is a natural number and $n \geq 1$.

Prove the statement $P(n)$: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ where n is a natural number and $n \geq 1$.

Prove the statement $P(n)$: $1 + 6 + 11 + \ldots + (5n - 4) = \frac{n(5n - 3)}{2}$, where n is a natural number, $n \geq 1$.