ANALYTIC SOLUTIONS FOR HARMONIC POTENTIALS INVOLVING CONCENTRIC LAYERED DIELECTRIC SPHERES

A Thesis

by

FRANK CISNEROS

BS, Texas A&M University - Corpus Christi, 2014

Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER of SCIENCE

in

MATHEMATICS

Texas A&M University-Corpus Christi
Corpus Christi, Texas

May 2018

ANALYTIC SOLUTIONS FOR HARMONIC POTENTIALS INVOLVING CONCENTRIC LAYERED DIELECTRIC SPHERES

A Thesis

by

FRANK CISNEROS

This thesis meets the standards for scope and quality of
Texas A&M University-Corpus Christi and is hereby approved.

Devanayagam Palaniappan, PhD
Chair

Alexey Sadovski, PhD
Committee Member

Beate Zimmer, PhD
Committee Member

May 2018

ABSTRACT

The mathematical problem of a conducting spherical core of radius a concentrically covered by a dielectric phase of radius b placed in an arbitrary external electric field is investigated. The vector field equations for the electric field (Maxwell equations) and the boundary conditions are transformed to a scalar boundary value problem (BVP) in terms of the harmonic potential functions. The harmonic potentials denoted by $\Phi^{\mathrm{I}}(r,\theta,\phi)$ and $\Phi^{\mathrm{II}}(r,\theta,\phi)$, where (r,θ,ϕ) are spherical coordinates, satisfy the Laplace equations in the regions $r > b$ and $a < r < b$, respectively. General analytical solutions for the potentials in the two phases are determined in infinite series form using spherical harmonics methods. Exact closed form solutions are also derived via an alternative approach. The latter solutions contain integrals involving harmonic functions. Our general solutions are applicable for arbitrary external potentials disturbed by a conducting spherical core with a dielectric coating. Several illustrative examples are investigated and exact solutions for them are constructed using our general solutions. The non-dimensional parameter $k = \frac{\varepsilon^{\mathrm{I}}}{\varepsilon^{\mathrm{I}}+\varepsilon^{\mathrm{II}}}$, where ε^{I} is the dielectric constant for the region $r > b$ and $\varepsilon^{\mathrm{II}}$ is the dielectric constant for the region $a < r < b$, influences the potential patterns in the case of externally imposed constant and linear fields. Our results for the source induced field indicate that the force is positive or negative depending on $k < 0.5$ or $k > 0.5$. Furthermore, the force is greater than zero when the core radius a approaches the value of the outer radius b. We believe that our mathematical results are of interest where coated dielectric objects are exposed to external electric fields.

ACKNOWLEDGEMENTS

My time here at Texas A&M University has been filled with entertaining moments. I am grateful to the instructors which encouraged me to push myself beyond my own limitations in both mathematics and computer science.

I am thankful to Dr. Pal for accepting me as his graduate student, for showing me how a graduate and higher level problems are approached. I want to thank my committee members Dr. Zimmer and Dr. Sadovski, for helping me to polish my thesis. Even though Dr. Denny was not on my committee, I want to thank her for a encouraging statement she told me one day. To the different chairs for the math department, Dr. Tintera, Dr. Mathur, and Dr. Sterba-Boatwright, I thank them for allowing me to complete my MS.

I gave thanks to my collages, Adam, Lane, Luis, Jesse to name a few for bowling to relax and take a step back on things. For the competitions we had to see whom received a higher grade on assignments and exams.

TABLE OF CONTENTS

CONTENTS PAGE

LIST OF FIGURES

LIST OF TABLES

CHAPTER I: INTRODUCTION

Electrostatic models involving dielectric layered conducting objects exposed to external electric fields have been widely used in a variety of physical, chemical, biological and medical fields. Applications of these models include interactions of dielectric coated surfaces in a host medium [8], behavior of multi-layered biological cells in applied electric fields [18], predicting the mobility of layer ions in electrolyte solutions [12], and the determination of surface potentials in electrocardiography [17]. Under simplified but practical conditions, the physical models lead to a mathematical boundary value problem for the governing field equations. The solutions of the boundary value problem require the construction of scalar potential functions satisfying the Laplace equation (a linear partial differential equation of second order), known as harmonic functions. The derivation of mathematical solutions for these models is challenging due to the mixed conditions at the bounding surfaces. In this thesis, we investigate an electrostatic model involving a dielectric coated conducting surface (spherical core) exposed to an applied electrostatic field.

Traditionally dielectric spherical models without a core were used for simplicity. This model treats the inside and outside of the single spherical interface with two dielectric constants assuming the continuity of the inner and outer electric fields at the boundary (see [13] for instance). However, in electrorheological fluids the active particles might consist of a conducting core material surrounded by a coating made of dielectric substance [5]. Thus the mathematical results which apply to concentrically coated geometries interacting with applied electrostatic fields are of significant importance. Recent studies on the mobility of multi-phase ion structures [12] and core-shell dielectric nanoparticles [8] highlight the growing interest in the subject.

A representative electrostatic model depicting a dielectric coated conducting spherical domain placed in another dielectric medium is shown in Figure 1.1. In this model a solid conducting inner sphere of radius a with an outer dielectric coated concentic layer with radius b and dielectric constant $\varepsilon^{\mathrm{II}}$ is placed in a host dielectric medium with a dielectric constant ε^{I} where an arbitrary electrostatic field is induced. Sample electric field representations in the respective phases $r > b$ and $a < r < b$ are indicated by the lines with arrows. It is evident that this scenario requires the knowledge of electrostatic fields in the respective phases of the relevant multiphase models. Theoretical and experimental results for models without either a dielectric layer or a conducting core are available in the literature [4, 7, 8]. A more recent investigation on the mathematical problem involving a dielectric sphere placed in an external electrostatic field in the context of magnetostatics addresses some of the challenges associated with multiphase problems [13]. Our primary goal in this thesis is to derive mathematical results for a multiphase problem with a concentric dielectric coated spherical domain interacting with arbitrary external electrostatic fields.

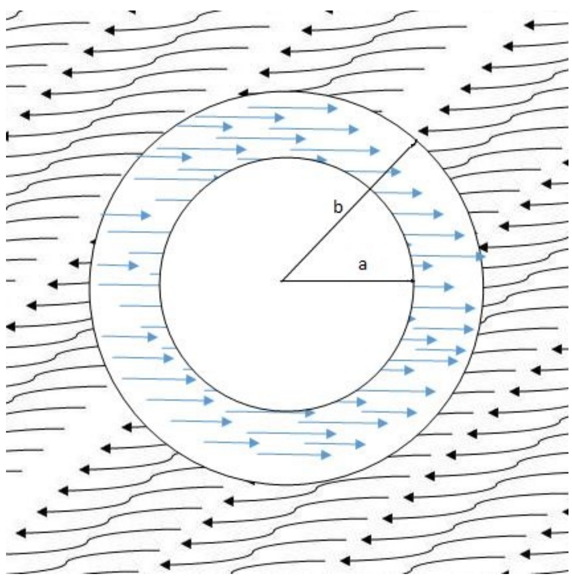

Figure 1.1
Electrostatic field patterns with a dielectric coated spherical core

The mathematical treatmeat of the model in question can be challenging and may require sophisticated mathematical tools. We use the Maxwell equations for the electrostatic fields in the two regions $r > b$ and $a < r < b$, which typically used in electrostatics [9, 14]. With the assumptions that the fields are time-independent (steady-state or static) the electric fields satisfy vector partial differential equations (PDEs) of second order. Solving the vector PDEs along with multiple boundary conditions can be cumbersome. Therefore, we transform the vector boundary value problem into a scalar boundary value problem using a transformation. This transformation introduces a scalar function, known as a *potential function* which satisfies the Laplace equation in each of the two regions. The solutions of the Laplace equation are known as *harmonic functions*. The determination of the harmonic potentials is essential in the study of electrical problems involving dielectric coated objects.

The scalar formulation discussed in the preceding paragraph has been used for two phase models. The problem of a dielectric/conducting sphere in an external host medium has been solved in many occasions [2, 4, 9, 13] using harmonic potentials formulation. The problem of a coated conducting sphere in an exterior field has also been treated with scalar potentials, but only for very special cases. For instance in [4], the problem of a coated conducting sphere placed in a constant electric field is solved (with less details than we will supply in Section 5.1). The problem of a layered conducting sphere in a source field is treated in separate studies [8, 16] without providing the calculation of forces acting on the boundary. For an arbitrary induced exterior field no general treatment has available before. Motivated by this fact we derive analytic solutions for the harmonic potentials involving a concentric dielectric layered conducting sphere placed in an arbitrary external electrostatic field.

In our derivation of the analytic solutions for the harmonic potentials we use spherical coordinates (r, θ, ϕ). First, we focus on the analytic solutions in an infinite series form using the spherical harmonics method [4]. This technique is very popular in dealing with electrostatic problems, especially with spherical boundaries. The sum of the infinite series expressions yield closed form solutions as shown in [13] for a dielectric sphere problem without a core. But this approach requires extensive efforts in the existence of an interior core. Therefore, we use an alternative approach to generate closed form solutions for the harmonic potentials. As will be shown later, this approach utilizes properties of solutions of the Laplace equation. Lastly, we apply our general solutions to some specific exterior fields and construct the field patterns. We believe that our mathematical results can be applied to practical situations where coated conducting spherical objects are of interest.

The organization of the thesis is as follows. In Chapter II, we provide the mathematical setting of our problem. The vector equations for the electric fields (Maxwell equations) and the corresponding scalar function formulation are discussed in Section 2.1 by introducing harmonic potentials. The boundary conditions for a dielectric coated conducting sphere, namely, the Dirichlet and Neumann conditions, are presented in the same section. For the sake of completeness, a short derivation of the solution of the Laplace equation in spherical coordinates is recorded in Section 2.2. The chapter concludes with some properties of the solutions of the Laplace equation illustrated in Section 2.3.

Our analytic solutions as infinite series for the harmonic potentials in the regions $r > b$ and $a < r < b$ are derived in Chapter III. The corresponding closed form solutions for the potentials involving a coated conducting spherical boundary are determined in Chapter IV. The special cases of our solutions in the electrostatics context are deduced. Chapter V contains the discussion of various examples for specific exterior potentials. The analytic solutions for the coated conducting sphere in a constant external field are provided in the Section 5.1. The equivalence of the solutions derived using the series representations and closed form expression is shown. Potential plots are supplied for some values of the parameters in the same section. Section 5.2 contains the corresponding results for a linear external field. We note that the solutions for the linear field in the presence of a layered conducting sphere have not been derived before. Analytic solutions for the harmonic potentials induced by a source field are found in Section 5.3. The force acting on the coated sphere is calculated in an infinite series form only for this example. The variation of the force with the parameters is illustrated by source plots. We point out that the results for the force due to a source field are new. Finally, the thesis concludes with the main findings in Chapter VI.

Below, we give a brief description of the spherical coordinate system and expressions for Legendre polynomials.

Spherical polar coordinate system: According to figure 1.2, using our notation, the conversion from Cartesian coordinates (x,y,z) to spherical coordinates (r,θ,ϕ) is as follows

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

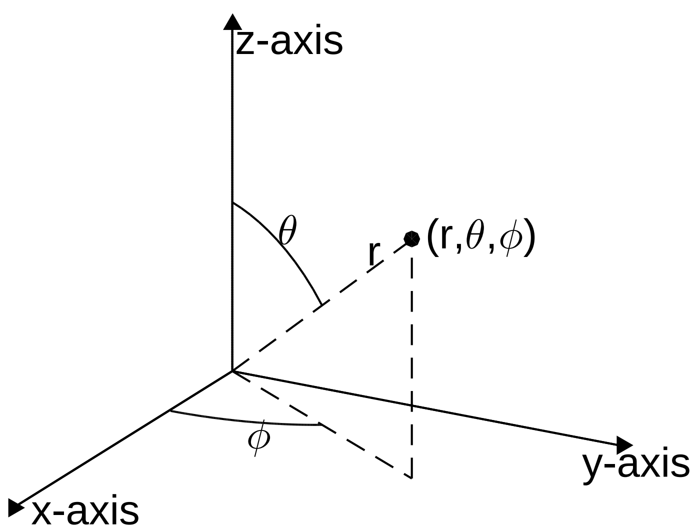

Figure 1.2
Spherical Coordinate System

Legendre polynomials: The associated Legendre polynomials are given by [4, 9]

$$P_\ell^m(x) = (-1)^m\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}\left[P_\ell(x)\right]$$

where

$$P_\ell(x) = \frac{m}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}\left[\left(x^2-1\right)^\ell\right].$$

In particular,

$$P_0^0(x) = 1$$

$$P_1^0(x) = x$$

$$P_1^1(x) = -1\left(1 - x^2\right)^{\frac{1}{2}}$$

$$P_2^0(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_2^1(x) = -3x\left(1 - x^2\right)^{\frac{1}{2}}$$

$$P_2^2(x) = 3\left(1 - x^2\right).$$

CHAPTER II: MATHEMATICAL SETTING

Mathematical problems in electric field theory have been studied since the time of Maxwell [7] and Kelvin [6]. There is now an extensive literature available on theoretical models involving bounded domains in the context of electrostatics [4, 7, 9]. Much attention has been devoted to the study of electrical properties of a solid (usually an isolated solid sphere) placed in an externally induced electric field governed by the Maxwell equations [7]. The simplified nature of the corresponding mathematical boundary value problem admits a detailed analysis for these simple models. When the solid in question is replaced by a dielectric spherical material, the problem needs more sophisticated mathematical tools as explained in [10, 18] and also in a more recent study [13]. The reason is that the relevant governing equations have to be solved subject to mixed boundary conditions on the boundary in question. If the existing boundary contains a complex structure, then the analysis has been considered very difficult. In fact, a systematic treatment of the mathematical problem in such scenarios has been ignored due to mathematical complexity. The main goal of this thesis is to address a unique approach leading to analytical solutions for a model consisting of a solid conducting sphere concentrically coated with a dielectric spherical inclusion placed in an arbitrary induced electric field. We use the Maxwell equations in the respective electric phases. Below, we describe the mathematical formulation of our model in detail. For the sake of completeness, we provide a short derivation of the general solution of the Laplace equation in spherical coordinates and a few properties pf this solution in the Section 2.3.

2.1 The Multiphase Problem

Consider a conducting sphere of radius a surrounded by a concentric dielectric layer of radius b and with dielectric constant $\varepsilon^{\mathrm{II}}$ placed in an external dielectric medium with dielectric constant ε^{I}, as shown in Figure 1.1. The electric field in the exterior phase is denoted by $\mathbf{E}^{\mathrm{I}}$ and the field in the phase with the dielectric layer is represented by $\mathbf{E}^{\mathrm{II}}$ such that the vector $\mathbf{E}$ is define as $\langle E_r, E_\theta, E_\phi \rangle$

in the respected phases. We take the model equations proposed by Maxwell [7] for the electric fields in the two regions. The vector field equations n a simplified from are given by [1, 4, 7]

$$\text{Exterior Domain} \quad r > b: \qquad \nabla^2 \mathbf{E}^{\mathrm{I}} = 0, \quad \nabla \cdot \mathbf{E}^{\mathrm{I}} = 0 \tag{2.1}$$

$$\text{Inside the Layer} \quad a < r < b: \qquad \nabla^2 \mathbf{E}^{\mathrm{II}} = 0, \quad \nabla \cdot \mathbf{E}^{\mathrm{II}} = 0 \tag{2.2}$$

where

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right] + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial}{\partial \theta}\right] + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}$$

∇^2 is the Laplace operator in $\mathbb{R}^3$, written in spherical coordinates (r, θ, ϕ), as illustrated in Figure 1.2. Equations (2.1) and (2.2) are the well-known Maxwell equations for the electric fields. Note that the first parts of (2.1) and (2.2) specify that both electric fields are vector harmonic functions, while the second parts state that the electric fields are divergence free.

The determine of the electric fields in the two phases requires solving the vector Laplace equations (first parts of (2.1) and (2.2)) subject to boundary conditions. This is a tedious effort and so we seek a scalar function formulation of our problem. Since the electric fields are irrotational [4, 7],

$$\nabla \times \mathbf{E}^{\mathrm{I}} = \nabla \times \mathbf{E}^{\mathrm{II}} = 0. \tag{2.3}$$

This implies according to [4] that there exists scalar functions Φ^{I} and Φ^{II} such that

$$\mathbf{E}^{\mathrm{I}} = -\nabla \Phi^{\mathrm{I}} \tag{2.4}$$

$$\mathbf{E}^{\mathrm{II}} = -\nabla \Phi^{\mathrm{II}}. \tag{2.5}$$

Now the application of the incompressibility (divergence free) condition of the electrical fields (second parts of (2.1) and (2.2)) to (2.4) and (2.5) yields

$$\nabla^2 \Phi^{\mathrm{I}} = 0 \tag{2.6}$$

$$\nabla^2 \Phi^{\mathrm{II}} = 0 \tag{2.7}$$

in the exterior and layered dielectric phases. Therefore, solving the vector equations (2.1) and (2.2) reduces to solving the scalar Laplace equations (2.6) and (2.7) for the potential functions subject to the given boundary conditions. The boundary conditions are taken to be [1, 4, 7]

- continuity of the tangential components of the electric fields on the outer sphere, that is, $E^{\mathrm{I}}_{\theta} = E^{\mathrm{II}}_{\theta}, \quad E^{\mathrm{I}}_{\phi} = E^{\mathrm{II}}_{\phi}$ on $r = b$
- continuity of the radial components of the electric displacement fields on the outer sphere, that is, $\varepsilon^{\mathrm{I}} E^{\mathrm{I}}_{r} = \varepsilon^{\mathrm{II}} E^{\mathrm{II}}_{r}$ on $r = b$.
- vanishing of the potential on the inner sphere, that is, $\Phi^{\mathrm{II}}_{r} = 0$ on $r = a$
- the potential and/or its derivatives go to a desired form at infinity

In terms of the electrostatic potentials the above conditions become

$$\Phi^{\mathrm{I}} = \Phi^{\mathrm{II}} \quad \text{on} \quad r = b \tag{2.8}$$

$$\varepsilon^{\mathrm{I}} \frac{\partial \Phi^{\mathrm{I}}}{\partial r} = \varepsilon^{\mathrm{II}} \frac{\partial \Phi^{\mathrm{II}}}{\partial r} \quad \text{on} \quad r = b \tag{2.9}$$

$$\Phi^{\mathrm{II}} = 0 \quad \text{on} \quad r = a \tag{2.10}$$

Note that equations (2.8) and (2.10) are Dirichlet type boundary conditions while (2.9) is a Neumann type boundary condition. Thus, the problem reduces down to solving the mixed boundary value problem for the Laplace equations (2.6) and (2.7) in the respective phases. The general solutions to the boundary value problem will yield the corresponding harmonic potentials Φ^{I} and Φ^{II} for the layered dielectric sphere model for an arbitrary induced potential. The determination of analytic solutions of our mixed boundary value problem for the Laplace equation is discussed in Chapters 3 and 4. In our analysis we will use a non-dimensional parameter k defined as

$$k = \frac{\varepsilon^{\mathrm{I}}}{\varepsilon^{\mathrm{I}} + \varepsilon^{\mathrm{II}}}. \tag{2.11}$$

In the next subsection we will provide a short derivation of the general solution of Laplace equation in spherical polar coordinates.

2.2 General Solution of the Laplace Equation in Spherical Coordinates

The Laplace's equation is stated as:

$$\nabla^2\Phi = 0. \tag{2.12}$$

Writing the in Laplace equation in spherical coordinates

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\Phi}{\partial r}\right] + \frac{1}{r^2\sin(\theta)}\frac{\partial\Phi}{\partial\theta}\left[\sin(\theta)\frac{\partial}{\partial\theta}\right] + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \tag{2.13}$$

Using separation of variables we are able to assume a solution of the form [15]

$$\Phi = R(r)G(\theta)H(\phi). \tag{2.14}$$

For simplicity we denote $R(r), G(\theta)$, and $H(\phi)$ as R, G, and H respectively then substituting (2.14) into (2.13) yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2R'GH\right] + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left[\sin(\theta)RG'H\right] + \frac{1}{r^2\sin^2(\theta)}RGH'' = 0. \tag{2.15}$$

By multiplying each term in (2.15) by r^2 and also dividing by RGH, we obtain

$$\frac{1}{R}\frac{d}{dr}\left[r^2R'\right] + \frac{1}{G\sin(\theta)}\frac{d}{d\theta}\left[\sin(\theta)G'\right] + \frac{1}{H\sin^2(\theta)}H'' = 0. \tag{2.16}$$

Equation (2.16) allows us to separate Laplace's equation into three separate ordinary differential equations; one a function of r, one a function of ϕ, with the last being a function of θ. Just as in separation of variables in Cartesian coordinates, we isolate terms that depend only on one variable. Because variables can be arbitrary values, we know that each of the equation must be equal to a constant. This means we can separate (2.16) into

$$\frac{H''}{H} = -m^2. \tag{2.17}$$

It can be shown that [15]

$$m^2 = \ell(\ell+1) \tag{2.18}$$

So we can set the next two components

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right) = \ell\left(\ell+1\right) \tag{2.19}$$

and

$$\frac{1}{G\sin\left(\theta\right)}\frac{d}{d\theta}\left(\sin\left(\theta\right)G'\right) = -\ell\left(\ell+1\right). \tag{2.20}$$

It might not seem trivial to pick this constant for (2.19) and (2.20) as it is for (2.17), but we know if we select it this way then we will produce a well known differential equation whose solution is already known. For one of the constants being negative (the angular part) while the other being positive (the radial part) is necessary so that the sum of equations is zero as required by Laplace's equation. Looking at (2.17) we get

$$H'' + m^2H = 0. \tag{2.21}$$

For (2.21) the separation constant is m^2, the equation takes the form of an harmonic equation which we get the following as a solution

$$H(\phi) = A_{\ell m}\sin(m\phi) + B_{\ell m}\cos(m\phi) \tag{2.22}$$

where $A_{\ell m}$ and $B_{\ell m}$ are constants. Taking the radial part, (2.19), multiply through by R and expanding the derivative.

$$r^2R'' + 2rR' - \ell(\ell+1)R = 0. \tag{2.23}$$

We can see that this is a Frobenius differential equation, and we know that the solution to the equation (2.23)

$$R(r) = C_\ell r^\ell + D_\ell r^{-(\ell+1)}, \tag{2.24}$$

where C_ℓ and D_ℓ are constants which will be determined once we apply our boundary equations.

Now taking the angular portion of the equation (2.20), we can perform the same methods as before multiply through by G and expanding the derivative.

$$G'' + \frac{\cos\theta}{\sin\theta} G' + \ell(\ell+1)G = 0. \tag{2.25}$$

This is the Legendre equation and its solutions are the Legendre polynomials of the second kind

$$G(\theta) = P_\ell^m(\cos\theta). \tag{2.26}$$

Substituting (2.22), (2.24) and (2.26) into (2.14) yields

$$\Phi = \left(Cr^\ell + Dr^{-(\ell+1)}\right) P_\ell^m(\cos\theta)(A_{\ell m}\sin(m\phi) + B_{\ell m}\cos(m\phi)). \tag{2.27}$$

By taking a sum of solutions of the form (2.27), we get our general solution to Laplace's equation in spherical coordinates. Switching ℓ to n for the form also that C and D will be absorbed into A and B results in

$$\Phi = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)}\right) P_n^m(\cos\theta)(\sin(m\phi) + \cos(m\phi)). \tag{2.28}$$

We will write (2.28) in the alternate form

$$\Phi = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)}\right) S_n(\theta,\phi) \tag{2.29}$$

where

$$S_n(\theta,\phi) = \sum_{m=0}^{\infty} P_n^m(\cos\theta)(\sin(m\phi) + \cos(m\phi)). \tag{2.30}$$

2.3 Various Forms of Solutions of the Laplace Equation

(i) Using Kelvin's Inversion we can show a solution to the Laplace equation [4].

If $\Phi_0(r,\theta,\phi)$ is a solution of the Laplace equation (2.12) then

$$\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right)$$

is also a solution of the Laplace equation.

Proof:

Let

$$\Phi_0(r,\theta,\phi) = \sum_{n=0}^{\infty} A_n r^n S_n(\theta,\phi) \tag{2.31}$$

So

$$\Phi_0\left(\frac{a^2}{r},\theta,\phi\right) = \sum_{n=0}^{\infty} A_n \left(\frac{a^2}{r}\right)^n S_n(\theta,\phi) \tag{2.32}$$

$$= \sum_{n=0}^{\infty} A_n \frac{a^{2n}}{r^n} S_n(\theta,\phi) \tag{2.33}$$

$$\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right) = \sum_{n=0}^{\infty} A_n \frac{a^{2n+1}}{r^{n+1}} S_n(\theta,\phi) \tag{2.34}$$

Therefore $\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right)$ is a solution of Laplace equation.

(ii) If $\Phi_0(r,\theta,\phi)$ is a solution of the Laplace equation (2.12) then

$$\int_0^1 \frac{a}{r}\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right) f(\lambda)\, d\lambda$$

where $f(\lambda)$ is a function independent of r,θ,ϕ also satisfies the Laplace equation.

Proof:

Let

$$\Phi_0(r,\theta,\phi) = \sum_{n=0}^{\infty} A_n r^n S_n(\theta,\phi) \tag{2.35}$$

So

$$\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right) = \sum_{n=0}^{\infty} A_n \left(\frac{\lambda a^2}{r}\right)^n S_n(\theta,\phi) \tag{2.36}$$

$$= \sum_{n=0}^{\infty} A_n \frac{\lambda^n a^{2n}}{r^n} S_n(\theta,\phi) \tag{2.37}$$

Now

$$\frac{a}{r}\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right)=\sum_{n=0}^{\infty}A_n\frac{a^{2n+1}}{r^{n+1}}\lambda^n S_n(\theta,\phi). \tag{2.38}$$

This implies, that

$$\int_0^1 \frac{a}{r}\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right)f(\lambda)d\lambda=\int_0^1\sum_{n=0}^{\infty}A_n\frac{a^{2n+1}}{r^{n+1}}\lambda^n S_n(\theta,\phi)d\lambda \tag{2.39}$$

$$=\sum_{n=0}^{\infty}A_n\frac{a^{2n+1}}{r^{n+1}}\frac{1}{n+1}S_n(\theta,\phi). \tag{2.40}$$

Therefore $\int_0^1 \frac{a}{r}\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right)f(\lambda)d\lambda$ is a solution of Laplace equation.

CHAPTER III: SERIES SOLUTIONS FOR THE MULTIPHASE MODEL

In this chapter we determine the analytical series solutions for the problem of a conducting core with a radius of a which is surrounded by another sphere with a radius of b having a dielectric constant $\varepsilon^{\mathrm{II}}$ that is placed in a dielectric field containing a dielectric constant ε^{I}. As described in Section 2.1, the problem reduces to solving the Laplace equation in two phases is subject to mixed boundary conditions (equations (2.8) - (2.10)). As mentioned before, a multiphase problem of this type has a variety of applications. It should be noted that the problem of a coated dielectric conducting sphere placed in a constant electric field was solved in [4]. However for a given arbitrary external field no general solutions seem to have been derived so far. In this chapter's section, we provide infinite series forms general solutions for the fields around a dielectric sphere with a concentric core placed in an externally induced field.

3.1 Potentials for the Multiphase Model in Series Form

There are various methods available to treat mixed boundary value problems for the Laplace equation [4, 7]. For problems involving spherical boundaries, the method of spherical harmonics has been used extersively. This method is based on expanding the solution of the Laplace equation in an infinite series form in terms of the spherical harmonic functions [7, 9]. The unknown coefficients are then determined via the boundary conditions. In the following, we show steps in the construction of the solution to our problem in the exterior model using the spherical harmonic expansion method.

Let $\Phi_0(r, \theta, \phi)$ be a given arbitrary potential satisfying the Laplace equation. This implies that $\Phi_0(r, \theta, \phi)$ is a harmonic function where (r, θ, ϕ) are the spherical coordinates as defined in Chapter

2. Now a harmonic function in spherical coordinates can be written in an infinite series form as [4]

$$\Phi_0\left(r,\theta,\phi\right)=\sum_{n=0}^{\infty}A_n r^n S_n\left(\theta,\phi\right) \tag{3.1}$$

where A_n is a constant coefficient and

$$S_n\left(\theta,\phi\right)=\sum_{m=0}^{\infty}P_n^m\left(\cos\theta\right)\left[\cos m\phi+\sin m\phi\right]. \tag{3.2}$$

The term $r^n S_n\left(\theta,\phi\right)$ is called a spherical harmonic of degree n and P_n^m are the Legendre Polynomials [15]. In the presence of a dielectric coated conducting sphere the problem has been formulated in Section 2.1 equations (2.8) - (2.10), into a boundary value problem. Below we derive the exact solutions of the boundary value problem for our model in infinite series forms.

Theorem 3.1: Let $\Phi_0\left(r,\theta,\phi\right)$ be an arbitrary potential field without the presence of a boundary. Assume a layered dielectric sphere of radius $r=b$, with a concentric core of radius $r=a$, centered at the origin $(0,0,0)$ is placed in the field of Φ_0. Then the potentials for regions I and II in series form are

$$\Phi^{\mathrm{I}}=\Phi_0\left(r,\theta,\phi\right)+\sum_{n=0}^{\infty}\left[\frac{(2k-1)\,nb^{2n+1}+(k-n-1)\,a^{2n+1}}{\Delta r^{n+1}}\right]A_n S_n\left(\theta,\phi\right) \tag{3.3}$$

$$\Phi^{\mathrm{II}}=\sum_{n=0}^{\infty}\left[\frac{(2n+1)\,k}{\Delta}r^n-\frac{(2n+1)\,ka^{2n+1}}{\Delta r^{n+1}}\right]A_n S_n\left(\theta,\phi\right) \tag{3.4}$$

where

$$\Delta=(k+n)+(n+1)\,(1-2k)\left(\frac{a}{b}\right)^{2n+1}. \tag{3.5}$$

Proof: Let $\Phi_0\left(r,\theta,\phi\right)$ be an arbitrary potential field without the presence of a boundary. Since Φ_0 is defined as harmonic function, it can be expressed in a series form as in (3.1) - (3.2). When a dielectric sphere, of radius b, with a concentric core, of radius a, is placed into the radical field of Φ_0, then the region I harmonic potential is a function and hence can be written as

$$\Phi^{\mathrm{I}}\left(r,\theta,\phi\right)=\sum_{n=0}^{\infty}\left[A_n r^n+\frac{B_n}{r^{n+1}}\right]S_n\left(\theta,\phi\right). \tag{3.6}$$

In region II the potential function satisfies the Laplace equation follows the same form as the previous notation

$$\Phi^{\mathrm{II}}(r,\theta,\phi)=\sum_{n=0}^{\infty}\left[C_n r^n+\frac{D_n}{r^{n+1}}\right]S_n(\theta,\phi). \tag{3.7}$$

The constant coefficients B_n, C_n, and D_n in (3.6) - (3.7) will be determined using the boundary conditions (2.8) - (2.10). Before we apply the boundary conditions, we can modify equation (2.9) to get the following

$$(1-k)\frac{\partial\Phi^{\mathrm{II}}}{\partial r}=k\frac{\partial\Phi^{\mathrm{I}}}{\partial r} \tag{3.8}$$

where k was defined in (2.11) as $k=\frac{\varepsilon^{\mathrm{I}}}{\varepsilon^{\mathrm{I}}+\varepsilon^{\mathrm{II}}}$ then

$$1-k=\frac{\varepsilon^{\mathrm{II}}}{\varepsilon^{\mathrm{II}}+\varepsilon^{\mathrm{I}}}. \tag{3.9}$$

Now by applying (3.6) - (3.7) to equations (2.8), to (2.10), use of (3.8) yields

$$A_n b^n+\frac{B_n}{b^{n+1}}=C_n b^n+\frac{D_n}{b^{n+1}} \tag{3.10}$$

$$(1-k)\left[nC_n b^{n-1}-(n+1)\frac{D_n}{b^{n+2}}\right]=k\left[nA_n b^{n-1}-(n+1)\frac{B_n}{b^{n+2}}\right] \tag{3.11}$$

$$C_n a^n+\frac{D_n}{a^{n+1}}=0 \tag{3.12}$$

Solving (3.12) for D_n:

$$D_n=-C_n a^{2n+1} \tag{3.13}$$

Substituting (3.13) into both (3.11) and (3.10) and also multiplying by b^{n+2} or b^{n+1}, respectively results in

$$(1-k)\left[n\left(b^{2n+1}+a^{2n+1}\right)+a^{2n+1}\right]C_n=k\left[nA_n b^{2n+1}-(n+1)B_n\right] \tag{3.14}$$

$$A_n b^{2n+1}+B_n=\left(b^{2n+1}-a^{2n+1}\right)C_n \tag{3.15}$$

Solve (3.15) for C_n:

$$C_n = \frac{A_n b^{2n+1} + B_n}{b^{2n+1} - a^{2n+1}}. \tag{3.16}$$

Now substitute (3.16) into (3.14). To avoid fractions we will also multiply by $\left(b^{2n+1} - a^{2n+1}\right)$

$$\begin{aligned}(1-k)\left[n\left(b^{2n+1}+a^{2n+1}\right)+a^{2n+1}\right]\left(A_n b^{2n+1}+B_n\right)\\ = k\left[nA_n b^{2n+1}-(n+1)B_n\right]\left(b^{2n+1}-a^{2n+1}\right).\end{aligned} \tag{3.17}$$

Solving for B_n in terms of A_n yields

$$B_n = \frac{kn\left(b^{2n+1}-a^{2n+1}\right)+(k-1)\left[n\left(b^{2n+1}+a^{2n+1}\right)+a^{2n+1}\right]}{k(n+1)\left(b^{2n+1}-a^{2n+1}\right)+(1-k)\left[n\left(b^{2n+1}+a^{2n+1}\right)+a^{2n+1}\right]} b^{2n+1} A_n. \tag{3.18}$$

After simplification, B_n can be written as

$$B_n = \frac{(2k-1)\,nb^{2n+1}+(k-n-1)\,a^{2n+1}}{(k+n)\,b^{2n+1}+(1-2k)\,(n+1)\,a^{2n+1}} b^{2n+1} A_n \tag{3.19}$$

B_n can be written in a compact form after divide the top and bottom of the expression by b^{2n+1} as

$$B_n = \frac{(2k-1)\,nb^{2n+1}+(k-n-1)\,a^{2n+1}}{\Delta} A_n \tag{3.20}$$

where Δ is defined as in equation (3.5). To solve for C_n, we substitute (3.19) into (3.16). This yields

$$C_n = \frac{A_n b^{2n+1}}{b^{2n+1}-a^{2n+1}}\left(1+\frac{(2k-1)\,nb^{2n+1}+(k-n-1)\,a^{2n+1}}{(k+n)\,b^{2n+1}+(1-2k)\,(n+1)\,a^{2n+1}}\right). \tag{3.21}$$

Simplifying (3.21) results in

$$C_n = \frac{(2n+1)\,k}{(k+n)\,b^{2n+1}+(1-2k)\,(n+1)\,a^{2n+1}} b^{2n+1} A_n. \tag{3.22}$$

In compact form it becomes

$$C_n = \frac{(2n+1)\,k}{\Delta} A_n. \tag{3.23}$$

Finally to get D_n, substitute (3.22) into (3.13):

$$D_n = -\frac{(2n+1)\,k}{(k+n)\,b^{2n+1}\,(1-2k)\,(n+1)\,a^{2n+1}}\,(ab)^{2n+1} A_n \tag{3.24}$$

Just as B_n and C_n, this can be written in compact form as

$$D_n = -\frac{(2n+1)\,k}{\Delta} a^{2n+1} A_n. \tag{3.25}$$

Now substituting (3.20), (3.23), and (3.25) back into equation (3.6) and (3.7) we get the series form solutions (3.3) - (3.5). ■

The series solutions given in Thorem 3.1 can be utilized to find analytic solutions for various externally imposed potentials $\Phi_0\,(r,\theta,\phi)$. We remark that it is also possible to determine a closed form solutions for our mixed boundary value problem given in (2.8) - (2.10). This may be achieved by finding the sum of the infinite series solutions given in (3.3) - (3.5) as done in [13], but we present an alternative approach which leads to exact closed form analytic solutions.

As stated in the previous chapter, the series solutions given in Theorem 3.1 can be summed to general closed form expressions for the potentials Φ^{I} and Φ^{II}. It is also possible to derive closed form solutions using a different approach. We demonstrate this approach in the following it is based on the properties of the solutions of the Laplace equation described in Section 2.3.

4.1 Potentials in Closed Form

It was shown in Section 2.3 that if $\Phi_0(r,\theta,\phi)$ is a solution of the Laplace equation then, so are $\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right)$ and $\int_0^1 \frac{a}{r}\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right)d\lambda$. The first is a solution of the Laplace equation by Kelvin's inversion property. The latter two is a solution via direct integration (see Section 2.3). Since the Laplace equation is linear, any linear combination of solutions is also a solution. We use these properties below to derive closed form solutions to the boundary value problem involving a dielectric coated conducting sphere placed in an arbitrary external field. We present out results in the form of a theorem.

Theorem 4.1: Let $\Phi_0\left(r,\theta,\phi\right)$ be an arbitrary potential field without the presence of a boundary. Assume a layered dielectric sphere of radius $r=b$, with a concentric core of radius $r=a$, centered at the origin $(0,0,0)$ is placed in the field of Φ_0. Then the potentials for regions I and II in closed form are

$$\Phi^{\mathrm{I}}\left(r,\theta,\phi\right)=\Phi_0\left(r,\theta,\phi\right)+\alpha\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right)-(1+\alpha)\frac{b}{r}\Phi_0\left(\frac{b^2}{r},\theta,\phi\right)$$
$$+\int_0^1\left[\frac{a}{r}\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right)-\frac{b}{r}\Phi_0\left(\frac{\lambda b^2}{r},\theta,\phi\right)\right]f\left(\lambda\right)d\lambda \qquad (4.1)$$

$$\Phi^{\mathrm{II}}(r,\theta,\phi) = \Phi_0(r,\theta,\phi) + \alpha\frac{a}{r}\Phi_0\left(\frac{a^2}{r},\theta,\phi\right) - (1+\alpha)\Phi_0(r,\theta,\phi) + \int_0^1\left[\frac{a}{r}\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right) - \Phi_0(\lambda r,\theta,\phi)\right]f(\lambda)\,d\lambda \tag{4.2}$$

where

$$\alpha = -\frac{2k}{1+(1-2k)\frac{a^3}{b^3}\frac{\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right)}{\Phi_0'(b,\theta,\phi)}}, \tag{4.3}$$

we define Φ_0' as the derivative of Φ respects to the first variable,

$$f(\lambda) = f(1)\,e^{(1-k)\int_\lambda^1 [G(y)]^{-1}dy} \tag{4.4}$$

$$G(\lambda) = \left(1+(1-2k)\frac{a}{b}\frac{\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)}{\Phi_0(\lambda b,\theta,\phi)}\right)\lambda \tag{4.5}$$

$$f(1) = -\frac{1+\alpha\left[1+\frac{(1-2k)}{k}\frac{a}{b}\frac{\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)}{\Phi_0(b,\theta,\phi)}\right]}{\left[1+(1-2k)\frac{a}{b}\frac{\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)}{\Phi_0(b,\theta,\phi)}\right]}k. \tag{4.6}$$

Proof: Guided by the solutions structure for a conducting and dielectric sphere in an electrostatic field [13], the forms for the potentials for our problem in question are chosen as given in (4.1) and (4.2). As shown in Section 2.3, all the terms in (4.1) and (4.2) are the solutions of the Laplace equation. We proceed now to find the constant α, and the function $f(\lambda)$ using the boundary conditions. Note that the chosen terms satisfy the Dirichlet conditions (2.8) and (2.10) on $r=b$ and $r=a$, respectively. The Neumann condition (2.9) on $r=b$ can be written in the form

$$\frac{\partial\Phi^{\mathrm{I}}}{\partial r} - \frac{(1-k)}{k}\frac{\partial\Phi^{\mathrm{II}}}{\partial r} = 0 \tag{4.7}$$

where k is defined as in (2.11). Application of (4.7) to the potentials given in (4.1) and (4.2) and setting $r = b$ leads to

$$
\begin{aligned}
&\Phi_0'(b,\theta,\phi) - \alpha\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right) - \alpha\frac{a^3}{b^3}\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right) + (1+\alpha)\frac{1}{b}\Phi_0(b,\theta,\phi) \\
&+ (1+\alpha)\Phi_0'(b,\theta,\phi) + \int\limits_0^1 \Big[-\frac{a}{b^2}\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right) - \frac{\lambda a^3}{b^3}\Phi_0'\left(\frac{\lambda a^2}{b},\theta,\phi\right) \\
&+ \frac{1}{b}\Phi_0(\lambda b,\theta,\phi) + \lambda\Phi_0'(\lambda b,\theta,\phi)\Big] f(\lambda)\,d\lambda - \frac{(1-k)}{k}\Bigg[\Phi_0'(b,\theta,\phi) \\
&- \alpha\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right) - \alpha\frac{a^3}{b^3}\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right) - (1+\alpha)\Phi_0'(b,\theta,\phi) \\
&+ \int\limits_0^1 \Big[-\frac{a}{b^2}\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right) - \frac{\lambda a^3}{b^3}\Phi_0'\left(\frac{\lambda a^2}{b},\theta,\phi\right) \\
&- \lambda\Phi_0'(\lambda b,\theta,\phi)\Big] f(\lambda)\,d\lambda\Bigg] = 0. \qquad (4.8)
\end{aligned}
$$

Various integrals appearing in (4.8) are simplified in the following way. The first two terms in the first integral in (4.8) can be rewritten as

$$
\begin{aligned}
&\int\limits_0^1 \left[-\frac{a}{b^2}\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right) - \frac{\lambda a^3}{b^3}\Phi_0'\left(\frac{\lambda a^2}{b},\theta,\phi\right)\right] f(\lambda)\,d\lambda \\
&\qquad = \int\limits_0^1 \left[-\frac{a}{b^2}\left[\frac{d}{d\lambda}\lambda\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)\right]\right] f(\lambda)\,d\lambda. \qquad (4.9)
\end{aligned}
$$

Using integration by parts and choosing

$$
\begin{aligned}
u &= -\frac{a}{b^2}f(\lambda) \qquad & dv &= \frac{d}{d\lambda}\left[\lambda\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)\right] \\
du &= -\frac{a}{b^2}f'(\lambda) \qquad & v &= \lambda\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right),
\end{aligned}
$$

we arrive at the following for the R.H.S of (4.9)

$$
-\frac{a}{b^2}f(\lambda)\lambda\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)\Bigg|_0^1 + \frac{a}{b^2}\int\limits_0^1 \lambda\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right) f'(\lambda)\,d\lambda.
$$

For the existence of the integrals, we assume

$$
\lim_{\lambda\to 0}\lambda f(\lambda) \to 0.
$$

Then the above expression becomes

$$-\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)f(1)+\frac{a}{b^2}\int_0^1\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)\lambda f'(\lambda)\,d\lambda. \tag{4.10}$$

Using the same approach for the other two terms in the first integral in (4.8) we find

$$\begin{aligned}\int_0^1\left[\frac{1}{b}\Phi_0(\lambda b,\theta,\phi)+\lambda\Phi_0'(\lambda b,\theta,\phi)\right]f(\lambda)\,d\lambda\\ =\int_0^1\frac{d}{d\lambda}\left[\frac{\lambda}{b}\Phi_0(\lambda b,\theta,\phi)\right]f(\lambda)\,d\lambda.\end{aligned} \tag{4.11}$$

Now integration by parts is used in the following way:

$$\begin{aligned}u&=f(\lambda) & dv&=\frac{d}{d\lambda}\left[\frac{\lambda}{b}\Phi_0(\lambda b,\theta,\phi)\right]\\ du&=f'(\lambda) & v&=\frac{\lambda}{b}\Phi_0(\lambda b,\theta,\phi),\end{aligned}$$

then the right hand side of (4.11) becomes

$$\frac{1}{b}\Phi_0(b,\theta,\phi)f(1)-\frac{1}{b}\int_0^1\lambda\Phi_0(\lambda b,\theta,\phi)f'(\lambda)\,d\lambda. \tag{4.12}$$

For the finally integral in (4.8)

$$\begin{aligned}\int_0^1-\lambda\Phi_0'(\lambda b,\theta,\phi)f(\lambda)\,d\lambda\\ =\int_0^1-\lambda\frac{d}{d\lambda}\left[\frac{1}{b}\Phi_0(\lambda b,\theta,\phi)\right]f(\lambda)\,d\lambda,\end{aligned} \tag{4.13}$$

the integration by parts is done as

$$\begin{aligned}u&=-\lambda f(\lambda) & dv&=\frac{d}{d\lambda}\left[\frac{1}{b}\Phi_0(\lambda b,\theta,\phi)\right]\\ du&=-\left[f(\lambda)+\lambda f'(\lambda)\right] & v&=\frac{1}{b}\Phi_0(\lambda b,\theta,\phi),\end{aligned} \tag{4.14}$$

resulting the following

$$-\frac{1}{b}\Phi_0(b,\theta,\phi)f(1)+\frac{1}{b}\int_0^1\Phi_0(\lambda b,\theta,\phi)\left[f(\lambda)+\lambda f'(\lambda)\right]d\lambda. \tag{4.15}$$

After the transformation for the integrals we are able to substitute the integrals in (4.8) with the terms from (4.10), (4.12), and (4.15) to get the following expression.

$$\begin{aligned}
&\Phi_0'(b,\theta,\phi) - \alpha\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right) - \alpha\frac{a^3}{b^3}\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right) + (1+\alpha)\frac{1}{b}\Phi_0(b,\theta,\phi) \\
&+ (1+\alpha)\Phi_0'(b,\theta,\phi) - \frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right) f(1) + \frac{a}{b^2}\int_0^1 \Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)\lambda f'(\lambda)\,d\lambda \\
&+ \frac{1}{b}\Phi_0(b,\theta,\phi) f(1) - \frac{1}{b}\int_0^1 \Phi_0(\lambda b,\theta,\phi)\lambda f'(\lambda)\,d\lambda - \frac{(1-k)}{k}\Bigg[\Phi_0'(b,\theta,\phi) \\
&- \alpha\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right) - \alpha\frac{a^3}{b^3}\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right) - (1+\alpha)\Phi_0'(b,\theta,\phi) \\
&- \frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right) f(1) + \frac{a}{b^2}\int_0^1 \Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)\lambda f'(\lambda)\,d\lambda - \frac{1}{b}\Phi_0(b,\theta,\phi) f(1) \\
&\quad + \frac{1}{b}\int \Phi_0(\lambda b,\theta,\phi)\left[f(\lambda) + \lambda f'(\lambda)\right]d\lambda\Bigg] = 0
\end{aligned} \tag{4.16}$$

Now α and $f(1)$ can be found by grouping the Φ_0 and Φ_0' terms separately. Afterwards the function $f(\lambda)$ is determined by collecting the terms in the integral.

First, setting the Φ_0' terms in (4.16) to zero, we get

$$\begin{aligned}
&\Phi_0'(b,\theta,\phi) - \alpha\frac{a^3}{b^3}\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right) + (1+\alpha)\Phi_0'(b,\theta,\phi) - \frac{(1-k)}{k}\Bigg[\Phi_0'(b,\theta,\phi) \\
&+ \alpha\frac{a^3}{b^3}\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right) + (1+\alpha)\Phi_0'(b,\theta,\phi)\Bigg] = 0
\end{aligned} \tag{4.17}$$

which yields

$$\alpha = -\frac{2k}{1 + (1-2k)\frac{a^3}{b^3}\frac{\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right)}{\Phi_0'(b,\theta,\phi)}}.$$

Next, setting the Φ_0 terms equal to zero, gives

$$\begin{aligned}
&-\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)+(1+\alpha)\frac{1}{b}\Phi_0(b,\theta,\phi)-\frac{a^2}{b}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)f(1)\\
&\quad\frac{1}{b}\Phi_0(b,\theta,\phi)f(1)+\frac{(1-k)}{k}\left[\alpha\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)\right.\\
&\quad\left.+\frac{a}{b^2}\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)f(1)+\frac{1}{b}\Phi_0(b,\theta,\phi)f(1)\right]=0
\end{aligned}\tag{4.18}$$

from which

$$f(1)=-\frac{1+\alpha\left[1+\frac{(1-2k)}{k}\frac{a}{b}\frac{\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)}{\Phi_0(b,\theta,\phi)}\right]}{\left[1+(1-2k)\frac{a}{b}\frac{\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)}{\Phi_0(b,\theta,\phi)}\right]}k.$$

Lastly, substituting these two terms in the integral leads to

$$\begin{aligned}
&\int_0^1\left[\left[\left(1-\frac{(1-k)}{k}\right)\frac{a}{b^2}\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)-\left(1+\frac{(1-k)}{k}\right)\frac{1}{b}\Phi_0(\lambda b,\theta,\phi)\right]\lambda f'(\lambda)\right.\\
&\quad\left.-\left[\frac{(1-k)}{k}\frac{1}{b}\Phi_0(\lambda b,\theta,\phi)\right]f(\lambda)\right]d\lambda=0.
\end{aligned}\tag{4.19}$$

By setting the intergrand in (4.19) to zero, we get a first order ordinary differential equation, namely,

$$M(\lambda)\lambda f'(\lambda)-N(\lambda)f(\lambda)=0\tag{4.20}$$

where

$$M(\lambda)=\left(1-\frac{(1-k)}{k}\right)\frac{a}{b^2}\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)-\left(1+\frac{(1-k)}{k}\right)\frac{1}{b}\Phi_0(\lambda b,\theta,\phi)\tag{4.21}$$

$$N(\lambda)=\frac{(1-k)}{k}\frac{1}{b}\Phi_0(\lambda b,\theta,\phi).\tag{4.22}$$

The solution to this first-order linear differential equation is given by

$$f(\lambda)=c_1e^{\int\frac{N(\lambda)}{M(\lambda)\lambda}d\lambda}.\tag{4.23}$$

Using some algebra, we can show that

$$\left[\frac{M(\lambda)}{N(\lambda)}\lambda\right]^{-1} = -(1-k)\left[\left[1+(1-2k)\frac{a}{b}\frac{\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)}{\Phi_0(\lambda b,\theta,\phi)}\right]\lambda\right]^{-1}. \tag{4.24}$$

Therefore

$$f(\lambda) = f(1)\, e^{(1-k)\int_{\lambda}^{1}[G(y)]^{-1}dy}$$

where

$$G(\lambda) = \left(1+(1-2k)\frac{a}{b}\frac{\Phi_0\left(\frac{\lambda a^2}{b},\theta,\phi\right)}{\Phi_0(\lambda b,\theta,\phi)}\right)\lambda.$$

This completes the derivation of the closed form solutions (4.1) and (4.2). ■

The equivalence of the series solutions (3.3) - (3.5) and the closed form solutions (4.1) - (4.6) is not obvious. By deriving exact solutions for specific external potentials, it can by shown that the two forms yield the same solutions. For constant and linear fields, this is illustrated in the following chapter.

We note that by setting $a = 0$ in (4.1) - (4.6) (or in (3.3) - (3.5)), we recover the case of a dielectric sphere placed in an arbitrary external field [13]. The conducting core is absent in this case. If we set $a = b$ in (4.1) - (4.6) and (3.3) - (3.5) then we obtain the well known results for a single conducting sphere placed in an arbitrary electrostatic field [4, 7, 9]. The dielectric coating is absent in this case.

CHAPTER V: EXAMPLES

In this chapter we will use the analytic solutions derived in Chapters 3 and 4 to determine exact solutions for some specific externally imposed potentials. The fields that we consider here are (i) a constant field; (ii) a linear field; and (iii) a field generated by a source. The expressions for the potentials $\Phi_0(r,\theta,\phi)$ for these induced fields are provided in table 4.1. For the constant and linear fields we derive solutions using our infinite series general solutions (3.3) - (3.5) as well as the closed form expressions (5.1) - (5.6) separately and show their equivalence. We provide analytic solutions for a monopole both in series and closed forms. We remark that the solutions for any given potentials (or electric field) can be derived in a similar fashion. We also calculate and discuss the force acting on the dielectric layered conducting sphere in the source field.

Potentials	$\Phi_0\,(r,\theta,\phi)$	E_{0x}	E_{0y}	E_{0z}
Constant	$Er\cos\theta$	0	0	E
Linear Field	$Hr^2 S_2(\theta,\phi)$	$\frac{4}{3}Hx$	$-\frac{2}{3}Hy$	$-\frac{2}{3}Hz$
Source	$\frac{m}{R_1}$	$-\frac{mx}{R_1}$	$-\frac{my}{R_1}$	$-\frac{m(z-c)}{R_1}$

Table 5.1
A set of different potentials which can be acting upon the multiphase model. Here E, H, and m are constants and $R_1^2 = r^2 - 2cr\cos\theta + c^2$ where c is the source location $(0,0,c)$.

5.1 Constant Field

As our first example, we consider a coated dielectric sphere with a concentric core is placed in a constant electric field. The potential $\Phi_0(r,\theta,\phi)$ in the absence of any boundaries is given by [4, 7] (see Table 4.1)

$$\Phi_0(r,\theta,\phi) = Er\cos(\theta) \tag{5.1}$$

This corresponds to $n = 1$ in (3.1) with $A_1 = E$. The analytical solutions in the two phases, $r > b$

and $a < r < b$ can be found using (3.3) and (3.4). By taking $n = 1$ in (3.3) - (3.5) we get

$$B_1 = \frac{(2k-1)\,b^3 + (k-2)\,a^3}{\Delta} E \tag{5.2}$$

$$C_1 = \frac{3k}{\Delta} E \tag{5.3}$$

$$D_1 = -\frac{3ka^3}{\Delta} E \tag{5.4}$$

where

$$\Delta = (k+1) + 2\,(1-2k)\left(\frac{a}{b}\right)^3 \tag{5.5}$$

Now the exact solutions for the potential functions in the respective phases are given by

$$\Phi^{\mathrm{I}}(r,\theta,\phi) = E\left[r + \left(\frac{(2k-1)\,b^3 + (k-2)\,a^3}{\Delta}\right)\frac{1}{r^2}\right]\cos(\theta) \qquad r > b \tag{5.6}$$

and

$$\Phi^{\mathrm{II}}(r,\theta,\phi) = E\left[\frac{3k}{\Delta}\left(r - \frac{a^3}{r^2}\right)\right]\cos(\theta) \qquad a < r < b \tag{5.7}$$

where Δ is defined as (5.5). The analytic solutions (5.6) - (5.8) can also be derived alternatively using our closed form expression (4.1) - (4.6) given in Theorem 4.1. We first determine α, $f(1)$, and $f(\lambda)$ appearing in (4.3) - (4.6). The constant α is found to be (4.3)

$$\alpha = -\frac{2k}{1 + (1-2k)\,\frac{a^3}{b^3}}. \tag{5.8}$$

To simplify (5.8) we let

$$\delta = 1 + (1-2k)\,\frac{a^3}{b^3}. \tag{5.9}$$

So α becomes

$$\alpha = -\frac{2k}{\delta} \tag{5.10}$$

The constant $f(1)$ is determined using (4.6)

$$f(1) = -\frac{1+\alpha\left[1+\frac{(1-2k)}{k}\frac{a^3}{b^3}\right]}{\delta}k \tag{5.11}$$

which in simplified form becomes

$$f(1) = -\frac{k\delta + 2k(k-1+\delta)}{\delta^2}. \tag{5.12}$$

The functions $G(\lambda)$ and $f(\lambda)$ are given by (using (4.4) - (4.5))

$$G(\lambda) = \delta\lambda \tag{5.13}$$

and

$$f(\lambda) = f(1)\, e^{\frac{(1-k)}{\delta}\int\limits_{\lambda}^{1}\frac{1}{y}dy}. \tag{5.14}$$

Evaluation of the integral leads to

$$f(\lambda) = f(1)\lambda^{-\frac{(1-k)}{\delta}}. \tag{5.15}$$

Using (5.12) in (5.15) we obtain

$$f(\lambda) = \left(-\frac{k\delta + 2k(k-1+\delta)}{\delta^2}\right)\lambda^{-\frac{(1-k)}{\delta}}. \tag{5.16}$$

Substituting in (5.10), (5.12), and (5.16) into (4.1) gives the solution for $r > b$ as

$$\begin{aligned}\Phi^{\mathrm{I}} = E\Bigg[& r + \left(-\frac{2k}{\delta}\right)\frac{a^3}{r^2} - \left(1-\frac{2k}{\delta}\right)\frac{b^3}{r^2} \\ &+ \left[\frac{a^3}{r^2} - \frac{b^3}{r^2}\right]\left(-\frac{k\delta + 2k(k-1+\delta)}{\delta^2}\right)\int\limits_0^1 \lambda^{\left(1-\frac{(1-k)}{\delta}\right)}d\lambda\Bigg]\cos(\theta).\end{aligned} \tag{5.17}$$

After integration we get

$$\begin{aligned}\Phi^{\mathrm{I}} = E\Bigg[& r + \frac{2k}{\delta r^2}\left(b^3 - a^3\right) - \frac{b^2}{r^2} \\ &+ \frac{1}{r^2}\left(a^3 - b^3\right)\left(-\frac{k\delta + 2k(k-1+\delta)}{\delta^2}\right)\left(\frac{\delta}{2\delta - (1-k)}\right)\Bigg]\cos(\theta).\end{aligned} \tag{5.18}$$

We can see that Δ be expressed in terms of δ as follows

$$\Delta = 2\delta - (1-k). \tag{5.19}$$

Now using (5.19) the expression for Φ^{I} becomes

$$\Phi^{\mathrm{I}} = E\left[r + \frac{2k}{\delta r^2}\left(b^3 - a^3\right) - \frac{b^3}{r^2} + \frac{1}{r^2\delta\Delta}\left(a^3 - b^3\right)\left(-k\delta + 2k\left(k-1+\delta\right)\right)\right]\cos(\theta). \tag{5.20}$$

Finally, we obtain

$$\Phi^{\mathrm{I}} = E\left[r + \left(\frac{(2k-1)\,b^3 + (k-2)\,a^3}{\Delta}\right)\frac{1}{r^2}\right]\cos(\theta) \qquad r > b. \tag{5.21}$$

Now for the potential Φ^{II} in the region $a < r < b$, with the α, $f(1)$, $f(\lambda)$ defined as in (5.10), (5.12), (5.16) we get from (4.2)

$$\Phi^{\mathrm{II}} = E\left[r - \frac{ak}{\delta}\frac{a^3}{r^2} - \left(1 - \frac{2k}{\delta}\right)r + \left[\frac{a^3}{r^2} - r\right]\int_0^1 f(1)\lambda^{1-\frac{(1-k)}{\delta}}\,d\lambda\right]\cos(\theta). \tag{5.22}$$

Evaluating the integral and using (5.19) we obtain

$$\Phi^{\mathrm{II}} = E\left[-\frac{2ka^3}{\delta r^2} + \frac{2k}{\delta}r + \left(\frac{a^3}{r^2} - r\right)\left(\frac{-k\delta + 2k\left(\Delta - \delta\right)}{\delta^2}\right)\left(\frac{\delta}{\Delta}\right)\right]\cos(\theta). \tag{5.23}$$

Equation (5.23) can be written in the form

$$\Phi^{\mathrm{II}} = E\left[\frac{a^3}{\delta\Delta r^3}\left(-2k\Delta - k\delta + 2k\left(\Delta-\delta\right)\right) + \frac{r}{\delta\Delta}\left(2k\Delta + k\delta - 2k\left(\Delta - \delta\right)\right)\right]\cos(\theta) \tag{5.24}$$

which after some algebra, simplifies to

$$\Phi^{\mathrm{II}} = E\left[\frac{3k}{\Delta}\left(r - \frac{a^3}{r^2}\right)\right]\cos(\theta) \qquad a < r < b. \tag{5.25}$$

The solutions given in (5.21) and (5.25) are precisely the same as those in (5.6) and (5.7) derived using infinite series expressions (3.3) - (3.5).

The potential field plots for the external constant field are shown in Figure 5.1 graphed using the exact solution in (5.21) and (5.25). The plots are shown in the xy-plane for different values of k $(0 \leq k \leq 1)$ and for a fixed $\frac{a}{b} = 0.5$. When $k = 0$, there is not potential in the dielectric phase (region II) as can be seen in Figure 5.1(a). In this case the problem reduces to that of a Dirichlet boundary value problem in classical electrostatics in [7, 13]. For $k > 0$, the field lines start appearing in region II. For small $k(= 0.25)$ the contours in region II appear to go around the core in a symmetrical form as in Figures 5.1(b). This scenario appears to exist for all $k > 0$ (Figure 5.1 (b) - (f)). But the contours get denser as the parameter k increases. The field lines in the phase I is also symmetrical about x-axis. Significant changes occur in phase I when $k > 0.5$ (see Figure 5.1(d) - (f)). For $k = 1.5$ the field lines in phase I appear to sharply penetrate the outer boundary $(r = b)$ vigorously. This particular situation is unphysical $\left(\varepsilon^{\mathrm{I}} < 0\right)$

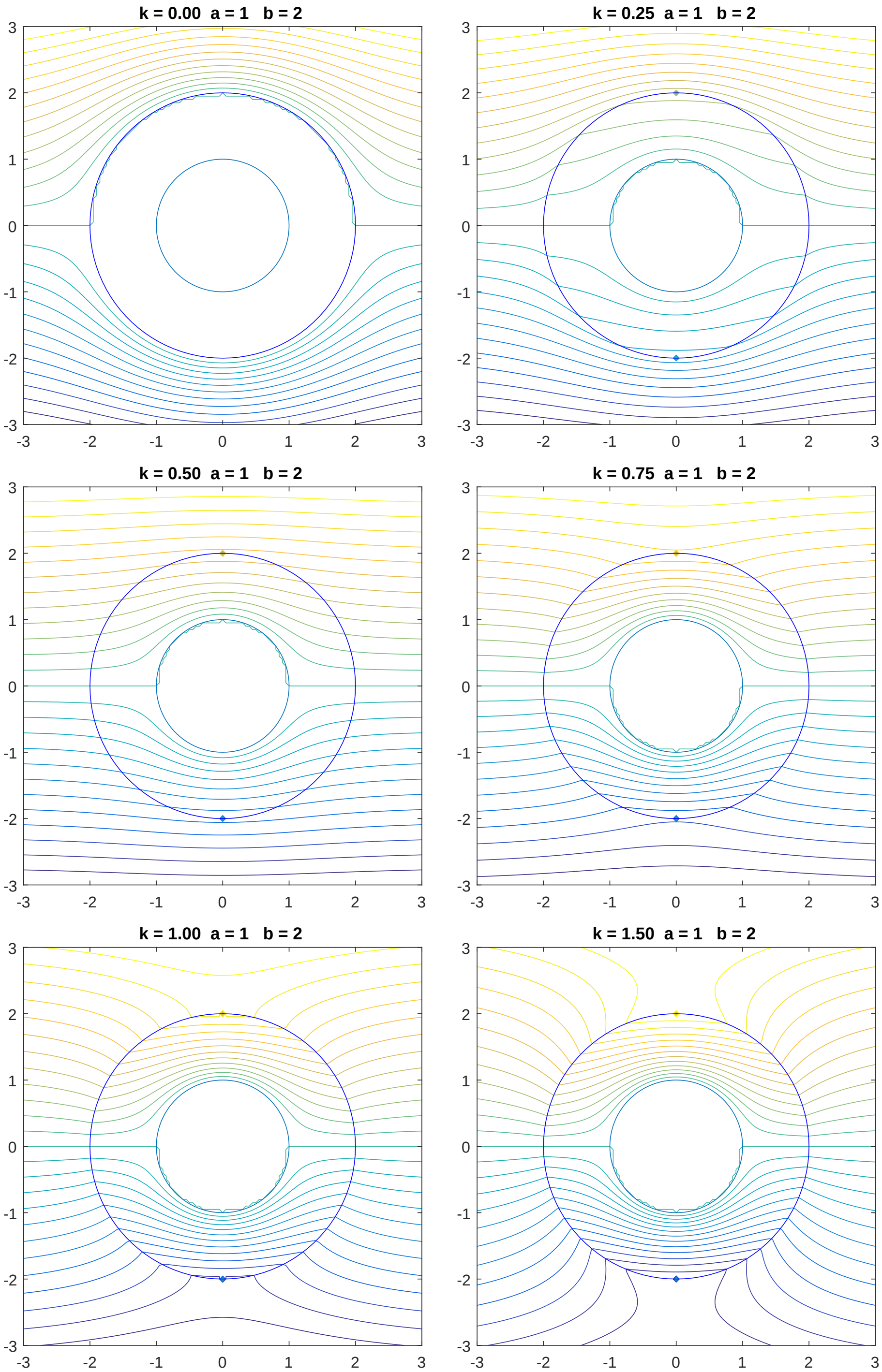

Figure 5.1
Potential plots for an inner sphere of radius 1 and outer sphere of radius 2 using the coated dielectric model for various of k: $k = 0$; $k = 0.25$; $k = 0.50$; $k = 0.75$; $k = 1.00$; $k = 1.50$ (unphysical)

5.2 Linear Dielectric Field

As our second example, we consider a insulated dielectric sphere with a core placed in a linear electric field. The potential harmonic functions $\Phi_0(r,\theta,\phi)$ without any boundaries is defined by [4, 7]

$$\Phi_0(r,\theta,\phi) = Hr^2 S_2(\theta,\phi). \tag{5.26}$$

This corresponds to $n = 2$ and $A_2 = H$ in (3.1). The exact solutions in the regions can be defined using (3.3) and (3.4). By taking $n = 2$ in (3.3) - (3.5) we get from the series form

$$\Phi^{\mathrm{I}}(r,\theta,\phi) = H\left[r^2 + \left(\frac{(4k-2)\,b^5 + (k-3)\,a^5}{\Delta}\right)\frac{1}{r^3}\right] S_2(\theta,\phi) \tag{5.27}$$

$$\Phi^{\mathrm{II}}(r,\theta,\phi) = H\left[\frac{5k}{\Delta}\left(r^2 - \frac{a^5}{r^3}\right)\right] S_2(\theta,\phi) \tag{5.28}$$

where

$$\Delta = (k+2) + 3\,(1-2k)\left(\frac{a}{b}\right)^5 \tag{5.29}$$

also $S_2(\theta,\phi)$ is defined as

$$S_2(\theta,\phi) = -P_2(\cos(\theta)) + \frac{1}{2}P_2^2(\cos\theta)\cos(2\phi) \tag{5.30}$$

that $P_2(\cos(\theta)) = \frac{1}{2}\left(3\cos^2(\theta) - 1\right)$ and $P_2^2(\cos(\theta)) = 3\sin^2(\theta)$. The exact solutions (5.27) - (5.29) can also be defined using the closed form expression (4.1) - (4.6). Using the potential $\Phi_0(r,\theta,\phi)$ as in (5.26). For the closed form, we define the constant terms α and $f(1)$ first

$$\alpha = -\frac{2k}{\delta} \tag{5.31}$$

where

$$\delta = 1 + (1-2k)\frac{a^5}{b^5} \tag{5.32}$$

using δ, (5.32), we are able to write Δ, (5.29), in terms of it

$$\Delta = k - 1 + 3\delta. \tag{5.33}$$

Expressing the second constant $f(1)$ using (4.6)

$$f(1) = \frac{-k\delta + 2k(\Delta - 2\delta)}{\delta^2}. \tag{5.34}$$

For the function terms, $G(\lambda)$ and $f(\lambda)$, we get

$$G(\lambda) = \delta\lambda. \tag{5.35}$$

Using (5.34) we are able to rewrite our secondary function $f(\lambda)$ as

$$f(\lambda) = \left(\frac{-k\delta + 2k(\Delta - 2\delta)}{\delta^2}\right)\lambda^{-\frac{(1-k)}{\delta}}. \tag{5.36}$$

Starting with Φ^{I} we are able to define it as using each term being defined as (5.31) - (5.36)

$$\begin{aligned}\Phi^{\mathrm{I}} = H\Bigg[r^2 - \frac{2k}{\delta}\frac{a^5}{r^3} - \left(1 - \frac{2k}{\delta}\right)\frac{b^5}{r^3} \\ + \left[\frac{a^5}{r^3} - \frac{b^5}{r^3}\right]\left(\frac{-k\delta + 2k(\Delta - 2\delta)}{\delta^2}\right)\int_0^1 \lambda^{2-\frac{(1-k)}{\delta}}d\lambda\Bigg]S_2(\theta,\phi).\end{aligned} \tag{5.37}$$

Evaluating the integral leads to

$$\Phi^{\mathrm{I}} = H\left[r^2 - \frac{2ka^5}{\delta r^3} - \frac{b^5}{r^3} + \frac{2kb^5}{\delta r^3} + \left[\frac{a^5}{r^3} - \frac{b^5}{r^3}\right]\left(\frac{-k\delta + 2k(\Delta - 2\delta)}{\delta^2}\right)\frac{1}{\Delta}\right]S_2(\theta,\phi). \tag{5.38}$$

Using some algebra, we are able to reduce (5.38) to

$$\Phi^{\mathrm{I}}(r,\theta,\phi) = H\left[r^2 + \left(\frac{(4k-2)b^5 + (k-3)a^5}{\Delta}\right)\frac{1}{r^3}\right]S_2(\theta,\phi) \qquad r < b. \tag{5.39}$$

For Φ^{II}, the same substitution as for Φ^{I} yields

$$\Phi^{\mathrm{II}} = H\left[\frac{2k}{\delta}r^2 - \frac{2ka^5}{\delta r^3} + \left[\frac{a^5}{r^3} - r^2\right]\left(\frac{-k\delta + 2k(\Delta - 2\delta)}{\delta^2}\right)\int_0^1 \lambda^{\left(2-\frac{(1-k)}{\delta}\right)}d\lambda\right]S_2(\theta,\phi). \tag{5.40}$$

Simplifying the expression and evaluating the integral we get

$$\Phi^{\mathrm{II}} = H\left[-\frac{2ka^5}{\delta r^3} + \frac{2k}{\delta}r^2 + \left[\frac{a^5}{\delta\Delta r^3} - \frac{1}{\delta\Delta}r^2\right](-k\delta + 2k(\Delta - 2\delta))\right]S_2(\theta,\phi). \tag{5.41}$$

Using some algebra we can express our solution in the reduced form

$$\Phi^{\mathrm{II}} = H\left[\frac{5k}{\Delta}\left(r^2 - \frac{a^5}{r^3}\right)\right]S_2(\theta,\phi) \qquad a < r < b. \tag{5.42}$$

The solutions given in (5.27) and (5.28) determined via the infinite series expressions are the same results as (5.39) and (5.42) derived using the closed form. The plots for the potentials Φ^{I} and Φ^{II} (plotted using (5.27) and (5.28)) for the linear field are shown in Figure 5.2. As in the previous example, we fix the radii ration $\frac{a}{b} = 0.5$ and change k. As shown in Figure 5.2(a). The plots for a Dirichlet problem are recovered when $k = 0$ [13]. It is clear that the inner core has no effect in this case. In general for all values of km the field lines in the exterior phase (region I) is symmetrical about both x and y axes. For $k \leq 0.5$, the contour lines are less denser (Figure 5.2(b) - (c)) compared to those for $k > 0.5$ (Figure 5.2(d) - (f))). When $k = 1.5$ (unphysical), more field lines get attracted towards the core in the dielectric coating, phase II. We note that the exact solutions (5.39) and (5.41) for the linear filed do not appear to have been reported earlier.

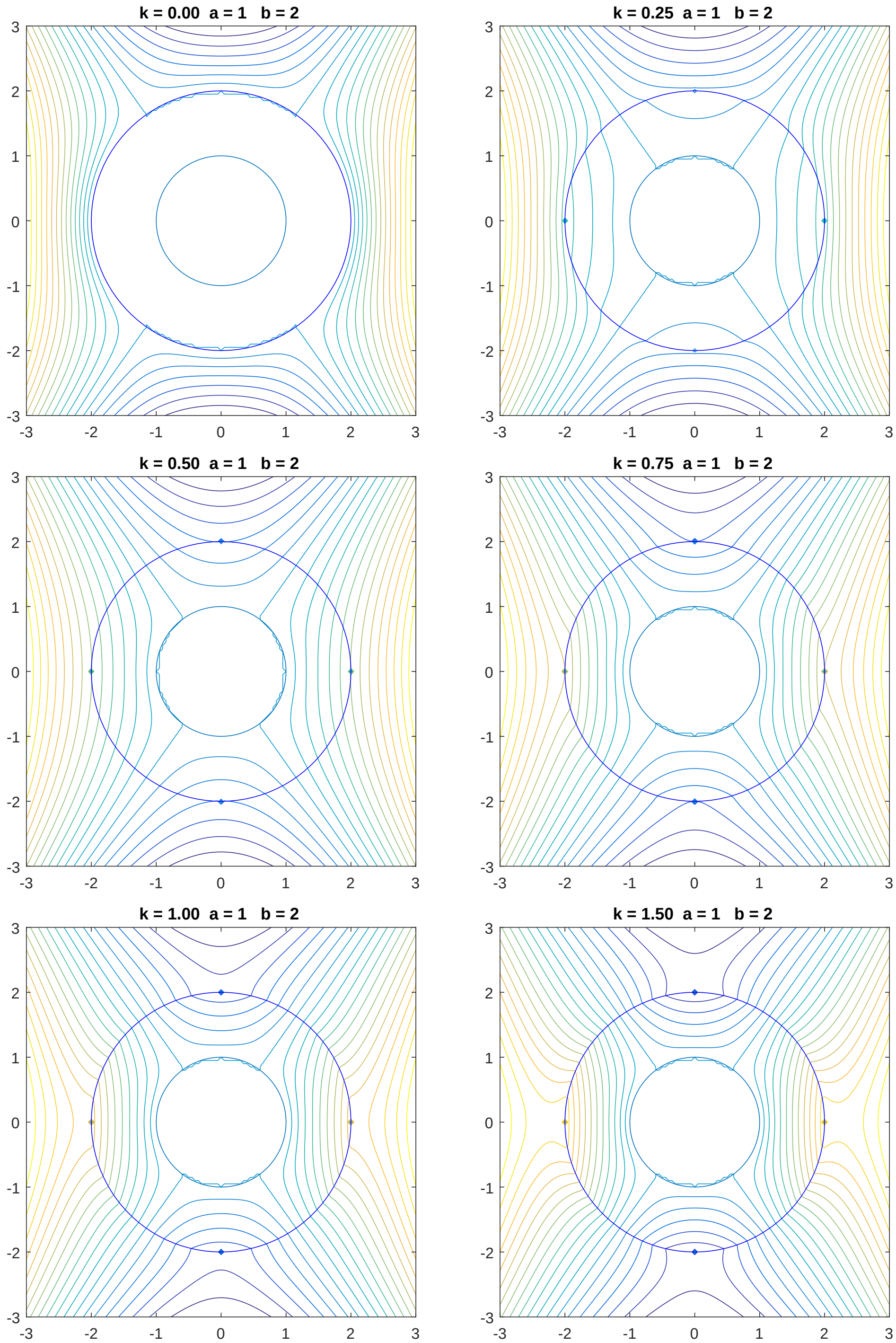

Figure 5.2
Potential plots for an inner sphere of radius 1 and outer sphere of radius 2 in the linear field using the multiphase model for various of k: $k = 0$; $k = 0.25$; $k = 0.50$; $k = 0.75$; $k = 1.00$; $k = 1.50$ (unphysical)

5.3 Electric Source Outside the Coated Conducting Sphere

Let us now consider an electric source of strength m located outside the concentrically dielectric coated conducting sphere at $(0,0,c)$, $c > b$. In the absence of the boundary, the solution to the Poisson equation is well-known [15]. It is given by

$$\Phi_0(r,\theta,\phi) = \frac{m}{\varepsilon^{\mathrm{I}}} \frac{1}{R_1(r)} \tag{5.43}$$

where

$$R_1(r) = \sqrt{r^2 - 2cr\cos\theta + c^2}.$$

When the dielectric coated conducting sphere is introduced into this source field, the modified potentials can be obtained in series form using (3.3) - (3.5). To this end, we first write

$$\Phi_0(r,\theta,\phi) = \frac{m}{\varepsilon^{\mathrm{I}}} \sum_{n=0}^{\infty} \frac{r^n}{c^{n+1}} P_n(cos\theta). \tag{5.44}$$

Comparison of (5.44) with (3.1) yields

$$A_n = \frac{m}{\varepsilon^{\mathrm{I}}} \frac{1}{c^{n+1}} \tag{5.45}$$

and $S_n(\theta,\phi) = P_n(\cos(\theta))$. Now using (3.3) - (3.4) we obtain the potentials in the exterior phase as

$$\Phi^{\mathrm{I}} = \frac{m}{\varepsilon^{\mathrm{I}}} \sum_{n=0}^{\infty} \left[r^n + \frac{(2k-1)\,nb^{2n+1} + (k-n-1)\,a^{2n+1}}{\Delta r^{n+1}} \right] \frac{1}{c^{n+1}} P_n(\cos(\theta)) \tag{5.46}$$

and in the interior phase

$$\Phi^{\mathrm{II}} = \frac{m}{\varepsilon^{\mathrm{I}}} \sum_{n=0}^{\infty} \left[\frac{(2n+1)\,k}{\Delta} r^n - \frac{(2n+1)\,ka^{2n+1}}{\Delta r^{n+1}} \right] \frac{1}{c^{n+1}} P_n(\cos(\theta)) \tag{5.47}$$

where Δ is defined as (3.5). To derive the closed form solutions using (4.1) - (4.6), we do the following: define $R_1(x_1,x_2)$ as

$$R(x_1,x_2) = \sqrt{x_1^2 - 2cx_1x_2\cos\theta + x_2^2c^2}. \tag{5.48}$$

It can then be shown that

$$\Phi_0\left(\frac{a^2}{r},\theta,\phi\right) = \frac{m}{\varepsilon^{\mathrm{I}}}\frac{1}{\sqrt{\left(\frac{a^2}{r}\right)^2 - 2c\left(\frac{a^2}{r}\right)\cos\theta + c^2}} \tag{5.49}$$

$$= \frac{m}{\varepsilon^{\mathrm{I}}}\frac{r}{\sqrt{a^4 - 2a^2cr\cos\theta + r^2c^2}} \tag{5.50}$$

$$= \frac{m}{\varepsilon^{\mathrm{I}}}\frac{r}{R(a^2,r)} \tag{5.51}$$

The other functions needed for the closed form solution in Theorem 4.1 can be constructed in a similar way and are given in Table 5.2.

$\Phi_0(r,\theta,\phi)$	Result	$\Phi_0(r,\theta,\phi)$	Result
$\Phi_0\left(\frac{b^2}{r},\theta,\phi\right)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{r}{R(b^2,r)}$	$\Phi_0\left(\frac{\lambda b^2}{r},\theta,\phi\right)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{r}{R(\lambda b^2,r)}$
$\Phi_0(\lambda r,\theta,\phi)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{r}{R(a^2,r)}$	$\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{r}{R(\lambda a^2,r)}$
$\Phi_0\left(\frac{a^2}{b},\theta,\phi\right)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{b}{R(a^2,b)}$	$\Phi_0(b,\theta,\phi)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{1}{R(b,1)}$
$\Phi_0\left(\frac{\lambda a^2}{r},\theta,\phi\right)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{b}{R(\lambda a^2,b)}$	$\Phi_0(\lambda b,\theta,\phi)$	$\frac{m}{\varepsilon^{\mathrm{I}}}\frac{r}{R(\lambda b,1)}$
$\Phi_0'\left(\frac{a^2}{b},\theta,\phi\right)$	$-\frac{m}{\varepsilon^{\mathrm{I}}}\frac{1}{R^2(a^2,b)}\left(a^2b - b^2c\cos\theta\right)$	$\Phi_0'(b,\theta,\phi)$	$-\frac{m}{\varepsilon^{\mathrm{I}}}\frac{1}{R^2(b,1)}(b - c\cos\theta)$

Table 5.2
The expression for all $\Phi_0(r,\theta,\phi)$ used in (4.1) - (4.2) in terms of (5.48)

Using the expressions given in Table 5.2, the constants and the functions appearing in (4.3) - (4.6) can be written in the following way:

$$f(1) = -\frac{k + \alpha\left[k + (1-2k)\frac{aR(b,1)}{R(a^2,b)}\right]}{1 + (1-2k)\frac{aR(b,1)}{R(a^2,b)}} \tag{5.52}$$

$$\alpha = -\frac{2kb^3R(a^2,b)(b - c\cos\theta)}{b^3R(a^2,b)(b - c\cos\theta) + (1-2k)a^3R(b,1)(a^2b - b^2c\cos\theta)} \tag{5.53}$$

$$G(\lambda) = \left(1 + (1-2k)\frac{aR(\lambda b,1)}{R(\lambda a^2,b)}\right)\lambda \tag{5.54}$$

$$f(\lambda) = f(1)\, e^{(1-k)\int_{\lambda}^{1}[G(y)]^{-1}dy}. \tag{5.55}$$

Now the exact solution found in (4.1) and (4.2) for the potentials for a dielectric coated conducting sphere placed in a source field become

$$\Phi^{\mathrm{I}}(r,\theta,\phi) = \frac{m}{\varepsilon^{\mathrm{I}}}\left[\frac{1}{R(r,1)} + \alpha\frac{a}{R(a^2,r)} - (1+\alpha)\frac{b}{R(b^2,r)} + \int_0^1 \left[\frac{a}{R(\lambda a^2,r)} - \frac{b}{R(\lambda b^2,r)}\right] f(\lambda)d\lambda\right] \quad \text{for} \quad r > b. \tag{5.56}$$

$$\Phi^{\mathrm{II}}(r,\theta,\phi) = \frac{m}{\varepsilon^{\mathrm{I}}}\left[\frac{1}{R(r,1)} + \alpha\frac{a}{R(a^2,r)} - (1+\alpha)\frac{1}{R(r,1)} + \int_0^1 \left[\frac{a}{R(\lambda a^2,r)} - \frac{r}{R(\lambda b,1)}\right] f(\lambda)d\lambda\right] \quad \text{for} \quad a < r < b. \tag{5.57}$$

The potentials functions in closed form given by (5.56) - (5.57) contain several integrals. The integrals can be evaluated in closed forms involving special functions or by using a computing software, but will not be presented here. Below we calculate the force acting on the coated sphere due to a source field. We will use the series solution (5.46) in our calculation.

5.4 Force due to a Source

The force $\mathbf{F}$ acting on a sphere of radius b due to a electric source of strength m located at $(x,y,z) = (0,0,c)$, where $c > b$ can be found according to [13] as

$$\mathbf{F} = m\left[\nabla\left(\Phi^{\mathrm{I}} - \Phi_0\right)\right]\Big|_{r=c}. \tag{5.58}$$

Here Φ^{I} is the exterior potential given in (5.46) and Φ_0 is the potential due to a source in the absence of a sphere given by (3.1). The subscript $r = c$ means the qualities are evaluated at $r = c, \theta = 0, \phi = 0$. Now the term $\Phi^{\mathrm{I}} - \Phi_0$ is

$$\Phi^{\mathrm{I}} - \Phi_0 = \sum_{n=0}^{\infty} \left[\frac{(2k-1)nb^{2n+1} + (k-n-1)a^{2n+1}}{\Delta r^{n+1}} \right] A_n P_n(\cos(\theta)) \tag{5.59}$$

where $A_n = \frac{1}{4\pi c^{n+1}}$. Taking the gradient of (5.59) and evaluating at $r = c, \theta = 0$ results in

$$\mathbf{F} = \sum_{n=0}^{\infty} \frac{(n+1)}{\Delta c^{2n+3}} \left[(1-2k)nb^{2n+1} + (1+n-k)a^{2n+1} \right] \hat{\mathbf{e}}_z \tag{5.60}$$

where $\hat{\mathbf{e}}_z$ is a unit vector in z-direction. We remark that the calculation of the force due to a source in the presence of a coated sphere given (5.60) has not been reported in the literature previously. As seen in (5.60), the force acts in the z-direction and depends on the radii, a and b, the dielectric parameter k and the location of the initial source c. The plots of the force are portrayed in Figure 5.3 using the conditions that $a = \beta b$ where $0 \leq \beta \leq 1$ to display each plot along the axes b^2Force vs. $\frac{b}{c}$.

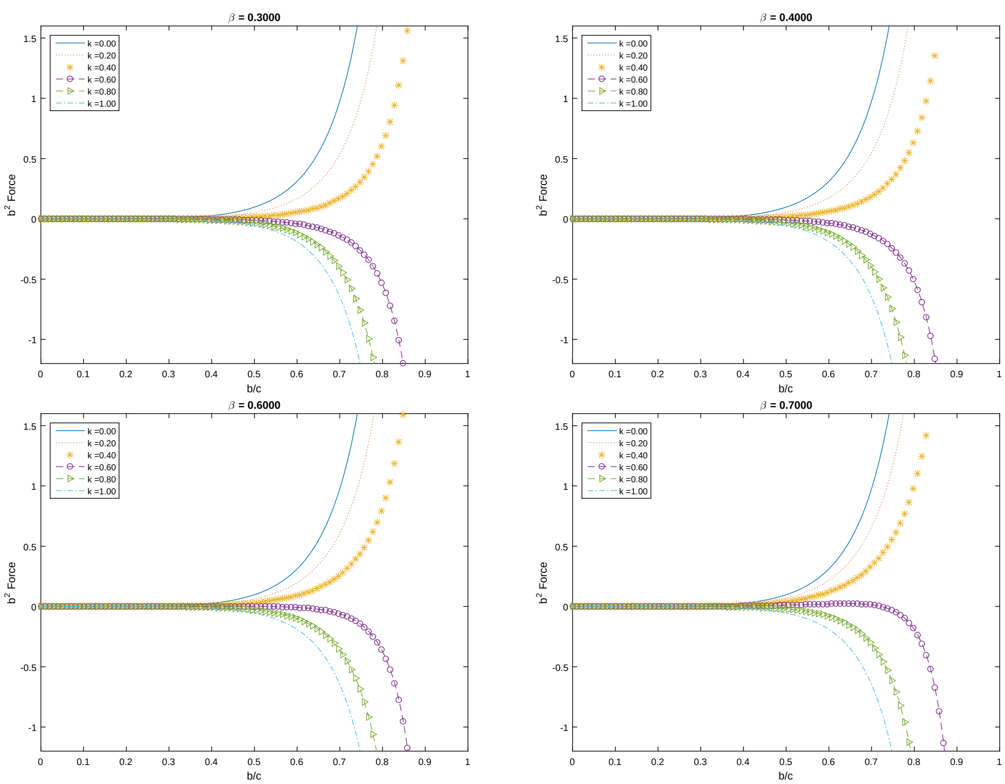

Figure 5.3
Plots of the force in a Source Field

From the illustrations in Figure 5.3 we can see that the magnitude of the force in the z-direction is positive and negative according to the values of k and β. Since we are using (5.60) to plot the force, we sum up the first 30 terms of the series (for convergence reasons). We are able to take the sum of more terms but the changes to the graph are minimal. In Figure 5.3(a) - (d) we notice that when $k < 0.5$ the force is increase but for $k > 0.5$ the force is decreasing according to the scale of the core proportional to the outer sphere. When the radius of the core approaches the size of outer sphere the force increase for all values of k ($0 \leq k \leq 1$).

CHAPTER VI: SUMMARY AND CONCLUSIONS

In this thesis, general solutions for the harmonic potentials for the mixed boundary value problem involving a dielectric coated conducting spherical core (radius a) placed in an externally induced arbitrary electrostatic field were derived. The potential functions in the exterior phase (region $r > b$, where b is the radius of the outer boundary) and in the dielectric phase (region $a < r < b$) satisfy mixed Dirichlet and Neumann type conditions. The analytic solutions in the two phases are developed as an infinite series (Chapter III) and also in closed form (Chapter IV). Our general solutions provide a mathematical scheme to generate exact mathematical results for the mixed boundary value problem involving layered conducting spherical boundaries.

Our general expressions for the potential functions $\Phi^{\mathrm{I}}(r,\theta,\phi)$ and $\Phi^{\mathrm{II}}(r,\theta,\phi)$ for the two phases were used to construct exact solutions for various externally imposed potentials. The induced potentials considered here were (i) a constant field; (ii) a linear field; and (iii) a source induced field (Chapter V). The equivalence of solutions derived using infinite series expressions (3.3) - (3.5) and closed form solutions (4.1) - (4.6) was shown for constant and linear fields. It is observed that the non-dimensional dielectric constant $k = \frac{\varepsilon^{\mathrm{I}}}{\varepsilon^{\mathrm{I}}+\varepsilon^{\mathrm{II}}}$ has a significant effect on the potential patterns. It is noted that for $k > 1$, the situation is unphysical because $\varepsilon^{\mathrm{II}} < 0$ in this case. For the source induced field in the presence of the coated conducting sphere, the force acting on the exterior boundary was calculated. The plots for the force reveals that the magnitude of the force is positive or negative according to whether $k > 0.5$ or $k < 0.5$. If the core radius approaches the exterior sphere radius then the force is positive for all k. Solutions to problems when the coated sphere is placed in dipole fields and other fields can be treated in the same manner.

The general infinite series and closed form solutions derived here for a given arbitrary external potential are new and have not been reported in the literature. Our approach to the problem with a coated conducting sphere provides a unique way of generating analytic solutions. The generalization of our approach to objects with multiple cores and also to non-spherical shapes will be the subject of future study.

REFERENCES

[1] A. Kirsch, and F. Hettlich, *The Mathematical Theory of Time-Harmonic Maxwell's Equations*, Springer , (2014)

[2] C. Trombley, and D. Palaniappan, *Interaction of a Straight Line Current with a Superconducting Sphere: Theoretical Results, J. Supercond. Nov. Magn.*, **29**, 89-100, (2016)

[3] D. Andelman, *Introduction to Electrostatics in Soft and Biological Matter*, Taylor & Francis, (2006)

[4] D. J. Griffiths, *Introduction to Electrodynamics*, 3rd Edn., Addison-Wesley, (1999)

[5] G. C. K. Yeh, J. Martinek, and G.S.S. Ludford., A General Sphere Theorem for Hydrodynamics, Heat, Magnetism, and Electrostatics, *Z. Angew Math. Mech.*, **36**, 111-116, (1956)

[6] H. Cavendish, *Multipoles; Macroscopic Media; Dielectrics*, Springer, (2000)

[7] J. C. A. Maxwell, *A Treatise on Electricity and Magnetism*, 2nd Edn. Vol. 1, Calendron Press, Oxford, (1881)

[8] J. C. E. Sten and I. V. Lindell, Electrostatic Image Theory for the Dielectric Sphere with an Internal Source, *Microwave and Optical Technology Letter*, **5**, 11, (1992)

[9] J. D. Jackson, *Classical Electrodynamics*, 3rd Edn., John Wiley & Sons, (1999)

[10] M. Ma, Z. Gan, and Z. Xu, Ion Structure Neat a Core-Shell Dielectric Nanoparticle, *Physical Rev. Lett.*, **118**, 076102, (2017)

[11] M. Stone, P.M. Goldbart, *Mathematics for Physics*, Pimander-Casaubon, (2008)

[12] R. A. Fouracre, M. J. Given, and B. H. Crichton, The Effect of Alternating Electric Fields on Ion Migration in Solid Dielectrics, *Journal of Physics C: Solid State Physics*, **19**, 12, (1949)

[13] R. Alarki, Calculation of Laplace and Helmholtz Potentials in Two-Phase Problems, MS Thesis, TAMUCC, (2017)

[14] R. Fitzpatrick, *Classical Electromagnetism*, 2nd Edition, (2001)

[15] T. Myint-U and L. Debnath, *Partial Differential Equations for Scientists and Engineers*, 4th Edition, Boston, Birkhauser, (2007)

[16] V. P. Pyati, Kelvin transformation for a dielectric-coated conducting sphere, *Radio Science*, Vol. 28, **6**, 1105-1110, (1993)

[17] Y. Petrov, An-isotropic spherical head model and its application to imaging electric activity of the brain, *Physical Review E*, **86**, (2012)

[18] Z. Xu, and W. Cai, Fast Analytical Methods for Macroscopic Electrostatic Models in Biomolecular Simulations, *SIAM Review*, **53**, 4, (2011)

NOMENCLATURE

(r, θ, ϕ)	spherical coordinates
$\mathbf{\Phi}^{\mathrm{I}}$	region I potential function
$\mathbf{\Phi}^{\mathrm{II}}$	region II potential function
$r > b$	region I
$b > r > a$	region II
ε^{I}	dielectric constant for region I
$\varepsilon^{\mathrm{II}}$	dielectric constant for region II
$\mathbf{E}^{\mathrm{I}}$	electric field for region I
$\mathbf{E}^{\mathrm{II}}$	electric field for region II
a	radius of core
b	radius of outer sphere